

The Complexity of Synchronizing Markov Decision Processes^{*,**}

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Abstract. We consider Markov decision processes (MDP) as generators of sequences of probability distributions over states. A probability distribution is p -synchronizing if the probability mass is at least p in a single state, or in a given set of states. We consider four temporal synchronizing modes: a sequence of probability distributions is always p -synchronizing, eventually p -synchronizing, weakly p -synchronizing, or strongly p -synchronizing if, respectively, all, some, infinitely many, or all but finitely many distributions in the sequence are p -synchronizing.

For each synchronizing mode, an MDP can be (i) *sure* winning if there is a strategy that produces a 1-synchronizing sequence; (ii) *almost-sure* winning if there is a strategy that produces a sequence that is, for all $\varepsilon > 0$, a $(1-\varepsilon)$ -synchronizing sequence; (iii) *limit-sure* winning if for all $\varepsilon > 0$, there is a strategy that produces a $(1-\varepsilon)$ -synchronizing sequence.

We provide fundamental results on the expressiveness, decidability, and complexity of synchronizing properties for MDPs. For each synchronizing mode, we consider the problem of deciding whether an MDP is sure, almost-sure, or limit-sure winning, and we establish matching upper and lower complexity bounds of the problems: for all winning modes, we show that the problems are PSPACE-complete for eventually and weakly synchronizing, and PTIME-complete for always and strongly synchronizing. We establish the memory requirement for winning strategies, and we show that all winning modes coincide for always synchronizing, and that the almost-sure and limit-sure winning modes coincide for weakly and strongly synchronizing.

1 Introduction

Markov decision processes (MDP) are finite-state stochastic models of dynamic systems studied in many applications such as planning [38], randomized algorithms [3,41], communication protocols [24], and in many problems related to reactive system design and verification [23,5,22]. MDPs exhibit both stochastic and nondeterministic behavior, as in the control problem for reactive systems: nondeterminism represents the possible choices of the controller, and stochasticity represents the uncertainties about the system response.

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The controller synthesis problem is to compute the largest probability with which a control strategy can ensure that the system satisfies a given specification, and to construct an optimal strategy [9,23]. The qualitative variant of the problem is to decide if the system can satisfy the specification with probability 1. Fundamental well-studied specifications are state-based and describe correct behaviors as infinite sequences of states of the MDP, including safety and liveness properties such as reachability, Büchi, and co-Büchi conditions, which require the system to visit a set of target states once, infinitely often, and ultimately always, respectively [29,18].

In contrast to this traditional approach, we consider a distribution-based semantics where the specification describes correct behaviors of MDPs as infinite sequences of probability distributions $d_i : Q \rightarrow [0, 1]$ over the finite state space Q of the system, where $d_i(q)$ is the probability that the MDP is in state $q \in Q$ after i execution steps. The distribution-based semantics is adequate in large-population models, such as systems biology [30], robot planning [7], distributed systems [26], etc. where the system consists of several copies of the same process (molecules, robots, sensors, etc.), and the relevant information along the execution of the system is the number of processes in each state, or the relative frequency (i.e., the probability) of each state. In the context of several identical processes, the same control strategy is used in every process, but the internal state of each process need not be the same along the execution, since probabilistic transitions may have a different outcome in each process. Therefore, the global execution of the system (consisting of all the processes) is better described by the sequence of probability distributions over states along the execution. However, the control strategy is local to each process and can select control actions depending on the full history of the process execution, which corresponds to general perfect-information strategies that we consider in this work.

Previously, the special case of blind strategies have been considered, which in each step select the same control action at all states, and thus only depend on the number of execution steps of the system. In automata theory, a blind strategy corresponds simply to an input word. In MDPs with blind strategies, also known as probabilistic automata [43,40], several basic problems are undecidable such as deciding if there exists a blind strategy that ensures a coBüchi condition with probability 1 [6], or deciding if a reachability condition can be ensured with probability arbitrarily close to 1 [27].

The main contribution of this paper is to establish the decidability and optimal complexity of deciding *synchronizing* properties for the distribution-based semantics of MDPs under general strategies. Synchronizing properties require that the sequence of probability distributions accumulate all the probability mass in a single state, or in a given set of states. They generalize synchronizing properties of finite automata [46,19]. Formally, for $0 \leq p \leq 1$ let a probability distribution $d : Q \rightarrow [0, 1]$ be *p-synchronized* if it assigns probability at least p to some state. A sequence $\bar{d} = d_0 d_1 \dots$ of probability distributions is

- (a) *always p-synchronizing* if d_i is *p-synchronized* for all i ;
- (b) *eventually p-synchronizing* if d_i is *p-synchronized* for some i ;
- (c) *weakly p-synchronizing* if d_i is *p-synchronized* for infinitely many i 's;
- (d) *strongly p-synchronizing* if d_i is *p-synchronized* for all but finitely many i 's.

The qualitative synchronizing properties, corresponding to the case where either $p = 1$, or p tends to 1, are analogous to the traditional safety, reachability, Büchi, and coBüchi conditions [17]. A typical application scenario of synchronizing properties is the design of a control program for a group of mobile robots running in a stochastic environment. The possible behaviors of the robots and the stochastic response of the environment (such as

Table 1. Winning modes for always, eventually, weakly, and strongly synchronizing objectives (where $\mathcal{M}_n^\alpha(T)$ denotes the probability that under strategy α , after n steps the MDP \mathcal{M} is in a state of T).

	Always			Eventually		
Sure	$\exists\alpha$	$\forall n$	$\mathcal{M}_n^\alpha(T) = 1$	$\exists\alpha$	$\exists n$	$\mathcal{M}_n^\alpha(T) = 1$
Almost-sure	$\exists\alpha$	\inf_n	$\mathcal{M}_n^\alpha(T) = 1$	$\exists\alpha$	\sup_n	$\mathcal{M}_n^\alpha(T) = 1$
Limit-sure	\sup_α	\inf_n	$\mathcal{M}_n^\alpha(T) = 1$	\sup_α	\sup_n	$\mathcal{M}_n^\alpha(T) = 1$
	Weakly			Strongly		
Sure	$\exists\alpha$	$\forall N \exists n \geq N$	$\mathcal{M}_n^\alpha(T) = 1$	$\exists\alpha$	$\exists N \forall n \geq N$	$\mathcal{M}_n^\alpha(T) = 1$
Almost-sure	$\exists\alpha$	$\limsup_{n \rightarrow \infty}$	$\mathcal{M}_n^\alpha(T) = 1$	$\exists\alpha$	$\liminf_{n \rightarrow \infty}$	$\mathcal{M}_n^\alpha(T) = 1$
Limit-sure	\sup_α	$\limsup_{n \rightarrow \infty}$	$\mathcal{M}_n^\alpha(T) = 1$	\sup_α	$\liminf_{n \rightarrow \infty}$	$\mathcal{M}_n^\alpha(T) = 1$

obstacle encounters) are represented by an MDP, and a synchronizing strategy corresponds to a control program that can be embedded in every robot to ensure that they meet (or synchronize) all the time, eventually once, infinitely often, or eventually forever.

We consider the following qualitative winning modes, summarized in Table 1: (i) *sure* winning, if there is a strategy that generates an {always, eventually, weakly, strongly} 1-synchronizing sequence; (ii) *almost-sure* winning, if there is a strategy that generates a sequence that is, for all $\varepsilon > 0$, {always, eventually, weakly, strongly} $(1 - \varepsilon)$ -synchronizing; (iii) *limit-sure* winning, if for all $\varepsilon > 0$, there is a strategy that generates an {always, eventually, weakly, strongly} $(1 - \varepsilon)$ -synchronizing sequence.

Contribution. The contributions of this paper are summarized as follows:

- *Expressiveness.* We show that the three winning modes form a strict hierarchy for eventually synchronizing: there are limit-sure winning MDPs that are not almost-sure winning, and there are almost-sure winning MDPs that are not sure winning. This is in contrast with the traditional state-based reachability objectives for which the notions of almost-sure and limit-sure winning coincide in MDPs. In this context, a more unexpected and difficult result is that the almost-sure and limit-sure modes coincide for weakly and strongly synchronizing. Thus those two synchronizing modes are more robust than eventually synchronizing, although we show that almost-sure weakly synchronizing strategies can be constructed from the analysis of eventually synchronizing (in limit-sure winning mode). Finally, for always synchronizing the three winning modes coincide, and we show that they coincide with a traditional safety objective.
- *Complexity.* For each synchronizing and winning mode, we consider the problem of deciding if a given initial distribution is winning. The complexity results are shown in Table 2 (p. 10). We establish the decidability and optimal complexity bounds for all winning modes. Under general strategies, the decision problems have much lower complexity than with blind strategies. We show that all decision problems are decidable, in polynomial time for always and strongly synchronizing, and PSPACE-complete for eventually and weakly synchronizing. This is also in contrast with almost-sure winning in the traditional semantics of MDPs, which is solvable in polynomial time for both safety and reachability [16].

- *Memory bounds.* We complete the picture by proving optimal memory bounds for winning strategies, summarized in Table 3 (p. 11). Memoryless strategies are sufficient for always synchronizing (like for safety objectives). We show that linear memory is sufficient for strongly synchronizing, and we identify a variant of strongly synchronizing for which memoryless strategies are sufficient. For eventually and weakly synchronizing, exponential memory is sufficient and may be necessary for sure winning strategies, and in general infinite memory is necessary for almost-sure winning.

Some results in this paper rely on insights about games and alternating automata that are of independent interest. Firstly, the sure-winning problem for eventually synchronizing is equivalent to a two-player game with a synchronized reachability objective, where the goal for the first player is to ensure that a target state is reached after a number of steps that is independent of the strategy of the opponent (and thus this number can be fixed in advance by the first player). This condition is stronger than plain reachability, and while the winner in two-player reachability games can be decided in polynomial time, deciding the winner for synchronized reachability is PSPACE-complete. This result is obtained by turning the synchronized reachability game into a one-letter alternating automaton for which the emptiness problem (i.e., deciding if there exists a word accepted by the automaton) is PSPACE-complete [32,34]. Secondly, our PSPACE lower bound for the limit-sure winning problem in eventually synchronizing uses a PSPACE-completeness result that we establish for the *universal finiteness problem*, which is to decide, given a one-letter alternating automata, whether from every state the accepted language is finite.

Related Works The traditional state-based semantics of MDPs has been studied extensively [42,15,23] and plays a central role in recent developments of system verification and controller synthesis, including expressiveness and complexity analysis of various classes of properties [25], using techniques such as symbolic algorithms for Büchi objectives [13], game-based abstraction techniques [35], and multi-objective analysis for assume-guarantee model-checking [22].

On the other hand, the distribution-based semantics has received a greater interest only recently, as it is shown that relevant key properties of MDPs can only be expressed in a distribution-based logical framework [8,37] and that a new useful notion of probabilistic bisimulation can be obtained in the distribution-based semantics [31]. Several recent works have investigated this new approach showing that the verification of quantitative properties of the distribution-based semantics is undecidable [37], and decidability can be obtained for special subclasses of systems [12], or through approximations [1]. In this context, a challenging goal is to identify useful decidable properties for the distribution-based semantics.

Synchronization problems were first considered for deterministic finite automata (DFA) where a *synchronizing word* is a finite sequence of control actions that can be executed from any state of an automaton and leads to the same state (see [46] for a survey of results and applications). While the existence of a synchronizing word can be decided in NLOGSPACE for DFA, extensive research effort is devoted to establishing a tight bound on the length of the shortest synchronizing word, which is conjectured to be $(n - 1)^2$ for automata with n states [11]. Various extensions of the notion of synchronizing word have been proposed for non-deterministic and probabilistic automata [10,33,36,20], leading to results of PSPACE-completeness [39], or even undecidability [36].

For probabilistic systems, it is natural to consider infinite input words (i.e., blind strategies) in order to study synchronization at the limit. In particular, almost-sure weakly and

strongly synchronizing with blind strategies has been studied [20] and the main result is that the problem of deciding the existence of a blind almost-sure winning strategy is undecidable for weakly synchronizing, and PSPACE-complete for strongly synchronizing [19,21]. In contrast, for general strategies, we establish the PSPACE-completeness and PTIME-completeness of deciding almost-sure weakly and strongly synchronizing respectively.

Note that while we solve the qualitative problems of synchronization for MDPs, the quantitative problem to decide, given a rational number $0 < p < 1$ whether an MDP is eventually p -synchronizing is likely to be hard since the Skolem problem (deciding if a linear recurrence sequence over the integers has a zero) reduces to it, even in the special case of Markov chains [2], and the decidability of the Skolem problem is a long-standing open question.

2 Markov Decision Processes and Synchronizing Properties

A *probability distribution* over a finite set S is a function $d : S \rightarrow [0,1]$ such that $\sum_{s \in S} d(s) = 1$. The *support* of d is the set $\text{Supp}(d) = \{s \in S \mid d(s) > 0\}$. We denote by $\mathcal{D}(S)$ the set of all probability distributions over S . Given a set $T \subseteq S$, let $d(T) = \sum_{s \in T} d(s)$ and $\|d\|_T = \max_{s \in T} d(s)$. For $T \neq \emptyset$, the *uniform distribution* on T assigns probability $\frac{1}{|T|}$ to every state in T . Given $s \in S$, the *Dirac distribution* on s assigns probability 1 to s , and by a slight abuse of notation we denote it simply by s .

A *Markov decision process* (MDP) is a tuple $\mathcal{M} = \langle Q, A, \delta \rangle$ where Q is a finite set of states, A is a finite set of actions, and $\delta : Q \times A \rightarrow \mathcal{D}(Q)$ is a probabilistic transition function. A state q is *absorbing* if $\delta(q, a)$ is the Dirac distribution on q for all actions $a \in A$. Given state $q \in Q$ and action $a \in A$, the successor state of q under action a is q' with probability $\delta(q, a)(q')$. Denote by $\text{post}(q, a)$ the set $\text{Supp}(\delta(q, a))$, and given $T \subseteq Q$ let $\text{Pre}(T) = \{q \in Q \mid \exists a \in A : \text{post}(q, a) \subseteq T\}$ be the set of states from which there is an action to ensure that the successor state is in T . For $k > 0$, let $\text{Pre}^k(T) = \text{Pre}(\text{Pre}^{k-1}(T))$ with $\text{Pre}^0(T) = T$. Note that the sequence $\text{Pre}^k(T)$ of iterated predecessors is ultimately periodic, precisely there exist $k < k' < 2^{|Q|}$ such that $\text{Pre}^k(T) = \text{Pre}^{k'}(T)$.

A *path* in \mathcal{M} is an infinite sequence $\pi = q_0 a_0 q_1 a_1 \dots$ such that $q_{i+1} \in \text{post}(q_i, a_i)$ for all $i \geq 0$. A finite prefix $\rho = q_0 a_0 q_1 a_1 \dots q_n$ of a path (or simply a finite path) has length $|\rho| = n$ and last state $\text{Last}(\rho) = q_n$. We denote by $\text{Path}(\mathcal{M})$ and $\text{Pref}(\mathcal{M})$ the set of all paths and finite paths in \mathcal{M} respectively.

For the decision problems considered in this paper, only the support of the probability distributions in the transition function is relevant (i.e., the exact value of the positive probabilities does not matter); therefore, we can assume that MDPs are encoded as A -labelled transition systems (Q, R) with $R \subseteq Q \times A \times Q$ such that $(q, a, q') \in R$ is a transition if $q' \in \text{post}(q, a)$.

Strategies A *randomized strategy* for \mathcal{M} (or simply a strategy) is a function $\alpha : \text{Pref}(\mathcal{M}) \rightarrow \mathcal{D}(A)$ that, given a finite path ρ , returns a probability distribution $\alpha(\rho)$ over the action set, used to select a successor state q' of ρ with probability $\sum_{a \in A} \alpha(\rho)(a) \cdot \delta(q, a)(q')$ where $q = \text{Last}(\rho)$.

A strategy α is *pure* if for all $\rho \in \text{Pref}(\mathcal{M})$, there exists an action $a \in A$ such that $\alpha(\rho)(a) = 1$; and *memoryless* if $\alpha(\rho) = \alpha(\rho')$ for all ρ, ρ' such that $\text{Last}(\rho) = \text{Last}(\rho')$. We view pure strategies as functions $\alpha : \text{Pref}(\mathcal{M}) \rightarrow A$, and memoryless strategies as functions $\alpha : Q \rightarrow \mathcal{D}(A)$.

Finally, a strategy α uses *finite-memory* if it can be represented by a finite-state transducer $T = \langle \text{Mem}, m_0, \alpha_u, \alpha_n \rangle$ where Mem is a finite set of modes (the memory of the strategy), $m_0 \in \text{Mem}$ is the initial mode, $\alpha_u : \text{Mem} \times (A \times Q) \rightarrow \text{Mem}$ is an update function that, given the current memory, last action, and state updates the memory, and $\alpha_n : \text{Mem} \times Q \rightarrow \mathcal{D}(A)$ is a next-move function that selects the probability distribution $\alpha_n(m, q)$ over actions when the current mode is m and the current state of \mathcal{M} is q . For pure strategies, we assume that $\alpha_n : \text{Mem} \times Q \rightarrow A$. The *memory size* of the strategy is the number $|\text{Mem}|$ of modes. For a finite-memory strategy α , let $\mathcal{M}(\alpha)$ be the Markov chain obtained as the product of \mathcal{M} with the transducer defining α .

2.1 State-based semantics

In the traditional state-based semantics, given an initial distribution $d_0 \in \mathcal{D}(Q)$ and a strategy α in an MDP \mathcal{M} , a *path-outcome* is a path $\pi = q_0 a_0 q_1 a_1 \dots$ in \mathcal{M} such that $q_0 \in \text{Supp}(d_0)$ and $a_i \in \text{Supp}(\alpha(q_0 a_0 \dots q_i))$ for all $i \geq 0$. The probability of a finite prefix $\rho = q_0 a_0 q_1 a_1 \dots q_n$ of π is

$$d_0(q_0) \cdot \prod_{j=0}^{n-1} \alpha(q_0 a_0 \dots q_j)(a_j) \cdot \delta(q_j, a_j)(q_{j+1}).$$

We denote by $\text{Outcome}(d_0, \alpha)$ the set of all path-outcomes from d_0 under strategy α . An *event* $\Omega \subseteq \text{Path}(\mathcal{M})$ is a measurable set of paths, and given an initial distribution d_0 and a strategy α , the probability $\text{Pr}^\alpha(\Omega)$ of Ω is uniquely defined [44]. We consider the following classical winning modes. Given an initial distribution d_0 and an event Ω , we say that \mathcal{M} is: *sure winning* if there exists a strategy α such that $\text{Outcome}(d_0, \alpha) \subseteq \Omega$; *almost-sure winning* if there exists a strategy α such that $\text{Pr}^\alpha(\Omega) = 1$; *limit-sure winning* if $\sup_\alpha \text{Pr}^\alpha(\Omega) = 1$, that is the event Ω can be realized with probability arbitrarily close to 1.

Given a set $T \subseteq Q$ of target states, and $k \in \mathbb{N}$, we denote by $\Box T = \{q_0 a_0 q_1 \dots \in \text{Path}(\mathcal{M}) \mid \forall i : q_i \in T\}$ the safety event of always staying in T , by $\Diamond T = \{q_0 a_0 q_1 \dots \in \text{Path}(\mathcal{M}) \mid \exists i : q_i \in T\}$ the event of reaching T , by $\Diamond^k T = \{q_0 a_0 q_1 \dots \in \text{Path}(\mathcal{M}) \mid q_k \in T\}$ the event of reaching T after exactly k steps, and by $\Diamond^{\leq k} T = \bigcup_{j \leq k} \Diamond^j T$ the event of reaching T within at most k steps. For example, if $\text{Pr}^\alpha(\Diamond T) = 1$ then almost-surely a state in T is reached under strategy α .

It is known for reachability objectives $\Diamond T$, that an MDP is almost-sure winning if and only if it is limit-sure winning, and the set of initial distributions for which an MDP is sure (resp., almost-sure or limit-sure) winning can be computed in polynomial time [16].

2.2 Distribution-based semantics

In contrast to the state-based semantics, we consider a symbolic outcome of MDPs viewed as generators of sequences of probability distributions over states [37]. Given an initial distribution $d_0 \in \mathcal{D}(Q)$ and a strategy α in \mathcal{M} , the *symbolic outcome* of \mathcal{M} from d_0 is the sequence $(\mathcal{M}_n^\alpha)_{n \in \mathbb{N}}$ of probability distributions defined by $\mathcal{M}_k^\alpha(q) = \text{Pr}^\alpha(\Diamond^k \{q\})$ for all $k \geq 0$ and $q \in Q$. Hence, \mathcal{M}_k^α is the probability distribution over states after k steps under strategy α . Note that $\mathcal{M}_0^\alpha = d_0$ and the symbolic outcome is a deterministic sequence of distributions: each distribution \mathcal{M}_k^α has a unique (deterministic) successor.

Informally, synchronizing objectives require that the probability of some state (or some group of states) tends to 1 in the sequence $(\mathcal{M}_n^\alpha)_{n \in \mathbb{N}}$, either always, once, infinitely often,

or always after some point. Given a set $T \subseteq Q$, consider the functions $sum_T : \mathcal{D}(Q) \rightarrow [0, 1]$ and $max_T : \mathcal{D}(Q) \rightarrow [0, 1]$ that compute $sum_T(d) = \sum_{q \in T} d(q)$ and $max_T(d) = \max_{q \in T} d(q)$. For $f \in \{sum_T, max_T\}$ and $p \in [0, 1]$, we say that a probability distribution d is p -synchronized according to f if $f(d) \geq p$, and that a sequence $\bar{d} = d_0 d_1 \dots$ of probability distributions is:

- (a) *always p -synchronizing* if d_i is p -synchronized for all $i \geq 0$;
- (b) *event (or eventually) p -synchronizing* if d_i is p -synchronized for some $i \geq 0$;
- (c) *weakly p -synchronizing* if d_i is p -synchronized for infinitely many i 's;
- (d) *strongly p -synchronizing* if d_i is p -synchronized for all but finitely many i 's.

For $p = 1$, these definitions are analogous to the traditional safety, reachability, Büchi, and coBüchi conditions [17], and we consider the following winning modes. Given an initial distribution d_0 and a function $f \in \{sum_T, max_T\}$, we say that for the objective of $\{\text{always, eventually, weakly, strongly}\}$ synchronizing from d_0 , the MDP \mathcal{M} is:

- *sure winning* if there exists a strategy α such that the symbolic outcome of α from d_0 is $\{\text{always, eventually, weakly, strongly}\}$ 1-synchronizing according to f ;
- *almost-sure winning* if there exists a strategy α such that for all $\varepsilon > 0$ the symbolic outcome of α from d_0 is $\{\text{always, eventually, weakly, strongly}\}$ $(1 - \varepsilon)$ -synchronizing according to f ;
- *limit-sure winning* if for all $\varepsilon > 0$, there exists a strategy α such that the symbolic outcome of α from d_0 is $\{\text{always, eventually, weakly, strongly}\}$ $(1 - \varepsilon)$ -synchronizing according to f ;

Note that the winning modes for synchronizing objectives differ from the traditional winning modes in MDPs: synchronizing objectives specify sequences of distributions, in a deterministic transition system with infinite state space (the states are the probability distributions). Since the transitions are deterministic and the probabilities are embedded in the state space, the behavior of the system is non-stochastic and the specification is simply a set of sequences (of distributions). In contrast, the traditional almost-sure and limit-sure winning modes of MDPs specify probability measures over sequences of states (called paths) in a probabilistic system with finite state space. Since the probabilities influence the transitions, the behavior of the system is stochastic and the specification is a set of probability measures over paths. For instance almost-sure reachability requires that the probability measure of *all* paths that visit a target state is 1, while almost-sure eventually synchronizing requires that the *single* symbolic outcome belongs to the set of sequences of distributions that are $(1 - \varepsilon)$ -synchronizing for all $\varepsilon > 0$.

We often write $\|d\|_T$ instead of $max_T(d)$ (and we omit the subscript when $T = Q$) and $d(T)$ instead of $sum_T(d)$, as in Table 1 where the definitions of the various winning modes and synchronizing objectives for $f = sum_T$ are summarized.

2.3 Membership problem

For $f \in \{sum_T, max_T\}$ and $\lambda \in \{\text{always, event, weakly, strongly}\}$, the *winning region* $\langle\langle 1 \rangle\rangle_{sure}^\lambda(f)$ is the set of initial distributions such that \mathcal{M} is sure winning for λ -synchronizing (we assume that \mathcal{M} is clear from the context). We define analogously the sets $\langle\langle 1 \rangle\rangle_{almost}^\lambda(f)$ and $\langle\langle 1 \rangle\rangle_{limit}^\lambda(f)$ of almost-sure and limit-sure winning distributions. For a singleton $T = \{q\}$ we have $sum_T = max_T$, and we simply write $\langle\langle 1 \rangle\rangle_\mu^\lambda(q)$ (where $\mu \in \{sure, almost, limit\}$).

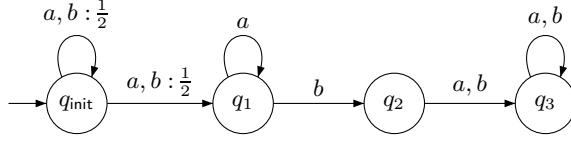


Fig. 1. An MDP such that $\langle\langle 1 \rangle\rangle_{sure}^\lambda(q_1) \neq \langle\langle 1 \rangle\rangle_{almost}^\lambda(q_1)$ for $\lambda \in \{event, weakly, strongly\}$, and such that $\langle\langle 1 \rangle\rangle_{almost}^{event}(q_2) \neq \langle\langle 1 \rangle\rangle_{limit}^{event}(q_2)$.

We are interested in the algorithmic complexity of the *membership problem*, which is to decide, given a probability distribution d_0 and a function f , whether $d_0 \in \langle\langle 1 \rangle\rangle_\mu^\lambda(f)$.

We show that the winning region is identical for always synchronizing in the three winning modes (Lemma 1), whereas for eventually synchronizing, the winning regions of the three winning modes are in general different (Lemma 2). First, note that it follows from the definitions that for all $f \in \{sum_T, max_T\}$, for all $\lambda \in \{always, event, weakly, strongly\}$, and all $\mu \in \{sure, almost, limit\}$:

- $\langle\langle 1 \rangle\rangle_\mu^{always}(f) \subseteq \langle\langle 1 \rangle\rangle_\mu^{strongly}(f) \subseteq \langle\langle 1 \rangle\rangle_\mu^{weakly}(f) \subseteq \langle\langle 1 \rangle\rangle_\mu^{event}(f)$, and
- $\langle\langle 1 \rangle\rangle_{sure}^\lambda(f) \subseteq \langle\langle 1 \rangle\rangle_{almost}^\lambda(f) \subseteq \langle\langle 1 \rangle\rangle_{limit}^\lambda(f)$.

Lemma 1. *Let T be a set of states. For all functions $f \in \{max_T, sum_T\}$, we have $\langle\langle 1 \rangle\rangle_{sure}^{always}(f) = \langle\langle 1 \rangle\rangle_{almost}^{always}(f) = \langle\langle 1 \rangle\rangle_{limit}^{always}(f)$.*

Proof. By the remark before the lemma, it suffices to show that $\langle\langle 1 \rangle\rangle_{limit}^{always}(f) \subseteq \langle\langle 1 \rangle\rangle_{sure}^{always}(f)$, that is for all distributions d_0 , if \mathcal{M} is limit-sure always synchronizing from d_0 , then \mathcal{M} is sure always synchronizing from d_0 . For $f = max_T$, consider ε smaller than the smallest positive probability in the initial distribution d_0 and in the transitions of the MDP $\mathcal{M} = \langle Q, A, \delta \rangle$. Then, given an always $(1 - \varepsilon)$ -synchronizing strategy, it is easy to show by induction on k that the distributions \mathcal{M}_k^α are Dirac for all $k \geq 0$. In particular d_0 is Dirac, and let $q_{init} \in T$ be such that $d_0(q_{init}) = 1$. It follows that there is an infinite path from q_{init} in the graph $\langle T, E \rangle$ where $(q, q') \in E$ if there exists an action $a \in A$ such that $\delta(q, a)(q') = 1$. The existence of this path entails that there is a loop reachable from q_{init} in the graph $\langle T, E \rangle$, and this naturally defines a sure-winning always synchronizing strategy in \mathcal{M} . A similar argument for $f = sum_T$ shows that for sufficiently small ε , an always $(1 - \varepsilon)$ -synchronizing strategy α must produce a sequence of distributions with support contained in T , until some support repeats in the sequence. This naturally induces an always 1-synchronizing strategy. \square

The results established in this article will entail that the almost-sure and limit-sure modes coincide for weakly and strongly synchronizing (see Theorem 7, Corollary 3, and Corollary 4). The other winning regions are distinct, as shown in the following lemma.

Lemma 2. *There exists an MDP \mathcal{M} and states q_1, q_2 such that:*

- (i) $\langle\langle 1 \rangle\rangle_{sure}^\lambda(q_1) \subsetneq \langle\langle 1 \rangle\rangle_{almost}^\lambda(q_1)$ for all $\lambda \in \{event, weakly, strongly\}$, and
- (ii) $\langle\langle 1 \rangle\rangle_{almost}^{event}(q_2) \subsetneq \langle\langle 1 \rangle\rangle_{limit}^{event}(q_2)$.

Proof. Consider the MDP \mathcal{M} with states q_{init}, q_1, q_2, q_3 and actions a, b as shown in Fig. 1. All transitions are deterministic except from q_{init} where on all actions, the successors are q_{init} and q_1 with probability $\frac{1}{2}$. Let q_{init} be the initial state.

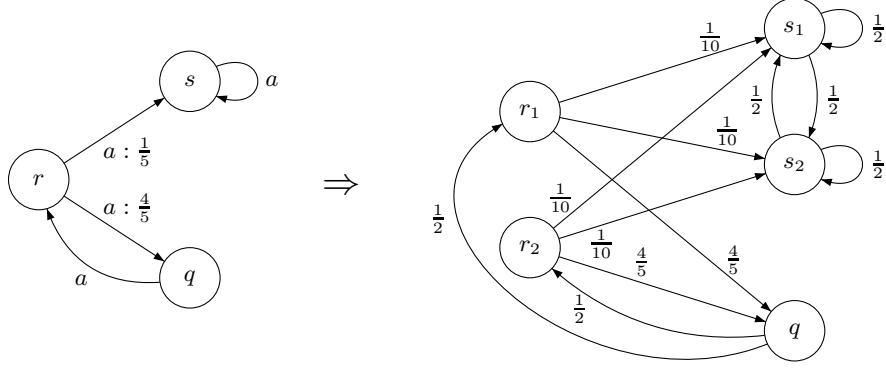


Fig. 2. State duplication ensures that the probability mass can never be accumulated in a single state except in q (we omit action a for readability).

To establish (i), by the remark before Lemma 1, it is sufficient to prove that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{strongly}}(q_1)$ and $q_{\text{init}} \notin \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(q_1)$, as it implies that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\lambda}(q_1)$ and $q_{\text{init}} \notin \langle\langle 1 \rangle\rangle_{\text{sure}}^{\lambda}(q_1)$ for all $\lambda \in \{\text{event}, \text{weakly}, \text{strongly}\}$. To prove that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{strongly}}(q_1)$, consider the pure strategy that always plays a . The outcome is such that the probability to be in q_1 after k steps is $1 - \frac{1}{2^k}$, showing that \mathcal{M} is almost-sure winning for the strongly synchronizing objective in q_1 (from q_{init}). On the other hand, $q_{\text{init}} \notin \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(q_1)$ because for all strategies α , the probability in q_{init} remains always positive, and thus in q_1 we have $\mathcal{M}_n^{\alpha}(q_1) < 1$ for all $n \geq 0$, showing that \mathcal{M} is not sure winning for the eventually synchronizing objective in q_1 (from q_{init}).

To establish (ii), for all $k \geq 0$ consider a strategy that plays a for k steps, and then plays b . Then the probability to be in q_2 after $k + 1$ steps is $1 - \frac{1}{2^k}$, showing that this strategy is eventually $(1 - \frac{1}{2^k})$ -synchronizing in q_2 . Hence, \mathcal{M} is limit-sure winning for the eventually synchronizing objective in q_2 (from q_{init}). Second, for all strategies, since the probability in q_{init} remains always positive, the probability in q_2 is always smaller than 1. Moreover, if the probability p in q_2 is positive after n steps ($p > 0$), then after any number $m > n$ of steps, the probability in q_2 is bounded by $1 - p < 1$. It follows that the probability in q_2 is never equal to 1 and cannot tend to 1 for $m \rightarrow \infty$, showing that \mathcal{M} is not almost-sure winning for the eventually synchronizing objective in q_2 (from q_{init}). \square

Finally, for eventually and weakly synchronizing we present in Lemma 3 a reduction of the membership problem with function \max_T to the membership problem with function $\text{sum}_{T'}$ for a singleton T' . It follows that the complexity results established in this article for eventually and weakly synchronizing with function sum_T also hold with function \max_T (this is trivial for the upper bounds, and for the lower bounds it follows from the fact that our hardness results hold for sum_T with singleton T , and thus for \max_T as well since in this case $\text{sum}_T = \max_T$).

Lemma 3. *For eventually and weakly synchronizing, in each winning mode the following problems are polynomial-time equivalent:*

- the membership problem with a function \max_T where T is an arbitrary subset of the state space, and

Table 2. Computational complexity of the membership problem.

	Always	Eventually	Weakly	Strongly
Sure	PTIME-C	PSPACE-C	PSPACE-C	PTIME-C
Almost-sure		PSPACE-C	PSPACE-C	PTIME-C
Limit-sure		PSPACE-C		

– the membership problem with a function $\text{sum}_{T'}$ where T' is a singleton.

Proof. Let $\mu \in \{\text{sure}, \text{almost}, \text{limit}\}$ and $\lambda \in \{\text{event}, \text{weakly}\}$. First we have $\langle\langle 1 \rangle\rangle_\mu^\lambda(\text{max}_T) = \bigcup_{q \in T} \langle\langle 1 \rangle\rangle_\mu^\lambda(q)$, showing that the membership problems for max and max_T are polynomial-time reducible to the corresponding membership problem for $\text{sum}_{T'}$ with singleton T' .

The reverse reduction is as follows. Given an MDP \mathcal{M} , a state q and an initial distribution d_0 , we can construct an MDP \mathcal{M}' and initial distribution d'_0 such that $d_0 \in \langle\langle 1 \rangle\rangle_\mu^\lambda(q)$ iff $d'_0 \in \langle\langle 1 \rangle\rangle_\mu^\lambda(\text{max}_{Q'})$ where Q' is the state space of \mathcal{M}' (thus $\text{max}_{Q'}$ is simply the function max). The idea is to construct \mathcal{M}' and d'_0 as a copy of \mathcal{M} and d_0 where all states except q are duplicated, and the initial and transition probabilities are equally distributed between the copies (see Fig. 2). Therefore if the probability tends to 1 in some state, it has to be in q . \square

The rest of this paper is devoted to the solution of the membership problem. It follows from the proof of Lemma 1 that the winning region for always synchronizing according to sum_T coincides with the set of winning initial distributions for the safety objective $\square T$ in the traditional semantics, which can be computed in polynomial time [14]. Moreover, always synchronizing according to max_T is equivalent to the existence of an infinite path staying in T in the transition system $\langle Q, R \rangle$ of the MDP restricted to transitions $(q, a, q') \in R$ such that $\delta(q, a)(q') = 1$, which can also be decided in polynomial time. In both cases, pure memoryless strategies are sufficient.

Theorem 1. *The membership problem for always synchronizing can be solved in polynomial time, and pure memoryless strategies are sufficient.*

For the other synchronizing modes (eventually, weakly, and strongly synchronizing), it is sufficient to consider Dirac initial distributions (i.e., assuming that MDPs have a single initial state) because the answer to the general membership problem for an MDP \mathcal{M} with initial distribution d_0 can be obtained by solving the membership problem for a copy of \mathcal{M} with a new initial state from which the successor distribution on all actions is d_0 .

In the next sections we present algorithms to decide the membership problem and we establish matching upper and lower bounds for the complexity of the problem: we show that eventually and weakly synchronizing are PSPACE-complete, whereas strongly synchronizing is PTIME-complete (like always synchronizing). Finally, we establish optimal memory bounds for the memory needed by strategies to win. Our results will also show that pure strategies are sufficient in all modes. The complexity results are summarized in Table 2, and we present the memory requirement for winning strategies in Table 3.

Table 3. Memory requirement.

	Always	Eventually	Weakly	Strongly	
				sum_T	max_T
Sure	memoryless	exponential	exponential	memoryless	linear
Almost-sure		infinite	infinite	memoryless	linear
Limit-sure		unbounded			

2.4 One-Letter Alternating Automata

In this section, we consider *one-letter alternating automata* (1L-AFA) as they have a structure of alternating graph analogous to MDP (i.e., when ignoring the probabilities). We review classical decision problems for 1L-AFA, and establish the complexity of a new problem, the *universal finiteness problem* which is to decide if from every initial state the language of a given 1L-AFA is finite. These results of independent interest are useful to establish the PSPACE lower bounds for eventually and weakly synchronizing in MDPs.

One-letter alternating automata Let $B^+(Q)$ be the set of positive Boolean formulas over Q , i.e. Boolean formulas built from elements in Q using \wedge and \vee . A set $S \subseteq Q$ satisfies a formula $\varphi \in B^+(Q)$ (denoted $S \models \varphi$) if φ is satisfied when replacing in φ the elements in S by **true**, and the elements in $Q \setminus S$ by **false**.

A *one-letter alternating finite automaton* is a tuple $\mathcal{A} = \langle Q, \delta_{\mathcal{A}}, \mathcal{F} \rangle$ where Q is a finite set of states, $\delta_{\mathcal{A}} : Q \rightarrow B^+(Q)$ is the transition function, and $\mathcal{F} \subseteq Q$ is the set of accepting states. We assume that the formulas in transition function are in disjunctive normal form. Note that the alphabet of the automaton is omitted, as it consists of a single letter. In the language of a 1L-AFA, only the length of words is relevant. For all $n \geq 0$, define the set $Acc_{\mathcal{A}}(n, \mathcal{F}) \subseteq Q$ of states from which the word of length n is accepted by \mathcal{A} as follows:

- $Acc_{\mathcal{A}}(0, \mathcal{F}) = \mathcal{F}$;
- $Acc_{\mathcal{A}}(n, \mathcal{F}) = \{q \in Q \mid Acc_{\mathcal{A}}(n-1, \mathcal{F}) \models \delta(q)\}$ for all $n > 0$.

For example, if $\delta(q_1) = (q_2 \wedge q_3) \vee q_4$ then the word of length n is accepted from q_1 if the word of length $n-1$ is accepted either from both q_2 and q_3 , or from q_4 . The set $\mathcal{L}(\mathcal{A}_q) = \{n \in \mathbb{N} \mid q \in Acc_{\mathcal{A}}(n, \mathcal{F})\}$ is the *language* accepted by \mathcal{A} from initial state q .

For fixed n , we view $Acc_{\mathcal{A}}(n, \cdot)$ as an operator on 2^Q that, given a set $\mathcal{F} \subseteq Q$ computes the set $Acc_{\mathcal{A}}(n, \mathcal{F})$. Note that $Acc_{\mathcal{A}}(n, \mathcal{F}) = Acc_{\mathcal{A}}(1, Acc_{\mathcal{A}}(n-1, \mathcal{F}))$ for all $n \geq 1$. Denote by $Pre_{\mathcal{A}}(\cdot)$ the operator $Acc_{\mathcal{A}}(1, \cdot)$. Then for all $n \geq 0$ the operator $Acc_{\mathcal{A}}(n, \cdot)$ coincides with $Pre_{\mathcal{A}}^n(\cdot)$, the n -th iterate of $Pre_{\mathcal{A}}(\cdot)$.

Decision problems We present classical decision problems for alternating automata, namely the emptiness and finiteness problems, and we introduce a variant of the finiteness problem that will be useful for solving synchronizing problems for MDPs.

- The *emptiness problem* for 1L-AFA is to decide, given a 1L-AFA \mathcal{A} and an initial state q , whether $\mathcal{L}(\mathcal{A}_q) = \emptyset$. The emptiness problem can be solved by checking whether $q \in Pre_{\mathcal{A}}^n(\mathcal{F})$ for some $n \geq 0$. It is known that the emptiness problem is PSPACE-complete, even for transition functions in disjunctive normal form [32,34].

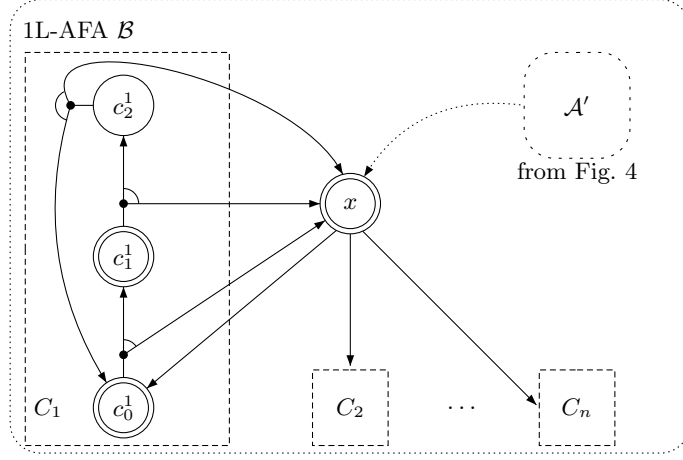


Fig. 3. Sketch of reduction to show PSPACE-hardness of the universal finiteness problem for 1L-AFA.

- The *finiteness problem* is to decide, given a 1L-AFA \mathcal{A} and an initial state q , whether $\mathcal{L}(\mathcal{A}_q)$ is finite. The finiteness problem can be solved in (N)PSPACE by guessing $n, k \leq 2^{|Q|}$ such that $\text{Pre}_{\mathcal{A}}^{n+k}(\mathcal{F}) = \text{Pre}_{\mathcal{A}}^n(\mathcal{F})$ and $q \in \text{Pre}_{\mathcal{A}}^n(\mathcal{F})$. The finiteness problem is PSPACE-complete by a simple reduction from the emptiness problem: from an instance (\mathcal{A}, q) of the emptiness problem, construct (\mathcal{A}', q') where $q' = q$ and $\mathcal{A}' = \langle Q, \delta', \mathcal{F} \rangle$ is a copy of $\mathcal{A} = \langle Q, \delta, \mathcal{F} \rangle$ with a self-loop on q (formally, $\delta'(q) = q \vee \delta(q)$ and $\delta'(r) = \delta(r)$ for all $r \in Q \setminus \{q\}$). It is easy to see that $\mathcal{L}(\mathcal{A}_q) = \emptyset$ iff $\mathcal{L}(\mathcal{A}'_{q'})$ is finite.
- The *universal finiteness problem* is to decide, given a 1L-AFA \mathcal{A} , whether $\mathcal{L}(\mathcal{A}_q)$ is finite for all states q . This problem can be solved by checking whether $\text{Pre}_{\mathcal{A}}^n(\mathcal{F}) = \emptyset$ for some $n \leq 2^{|Q|}$, and thus it is in PSPACE. Note that if $\text{Pre}_{\mathcal{A}}^n(\mathcal{F}) = \emptyset$, then $\text{Pre}_{\mathcal{A}}^m(\mathcal{F}) = \emptyset$ for all $m \geq n$.

Given the PSPACE-hardness proofs of the emptiness and finiteness problems, it is not easy to see that the universal finiteness problem is PSPACE-hard.

Lemma 4. *The universal finiteness problem for 1L-AFA is PSPACE-hard.*

Proof. We show the result by a reduction from the emptiness problem for 1L-AFA, which is PSPACE-complete [32,34]. The language of a 1L-AFA $\mathcal{A} = \langle Q, \delta, \mathcal{F} \rangle$ from initial state q_0 is non-empty if $q_0 \in \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for some $i \geq 0$. Since the sequence $\text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ is ultimately periodic, it is sufficient to compute $\text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for every $i \leq 2^{|Q|}$ to decide emptiness.

From \mathcal{A} , we construct a 1L-AFA $B = \langle Q', \delta', \mathcal{F}' \rangle$ with set \mathcal{F}' of accepting states such that the sequence $\text{Pre}_B^i(\mathcal{F}')$ in B mimics the sequence $\text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ in \mathcal{A} for $2^{|Q|}$ steps. The automaton B contains the state space of \mathcal{A} , i.e. $Q \subseteq Q'$. The goal is to have $\text{Pre}_B^i(\mathcal{F}') \cap Q = \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for all $i \leq 2^{|Q|}$, as long as $q_0 \notin \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$. Moreover, if $q_0 \in \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for some $i \geq 0$, then $\text{Pre}_B^j(\mathcal{F}')$ will contain q_0 for all $j \geq i$ (the state q_0 has a self-loop in B), and if $q_0 \notin \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for all $i \geq 0$, then B is constructed such that $\text{Pre}_B^j(\mathcal{F}') = \emptyset$ for sufficiently large j (roughly for $j > 2^{|Q|}$). Hence, the language of \mathcal{A} is non-empty if and only if the sequence $\text{Pre}_B^j(\mathcal{F}')$ is not ultimately empty, that is if and only if the language of B is infinite from some state (namely q_0).

The key is to let B simulate \mathcal{A} for exponentially many steps, and to ensure that the simulation stops if and only if q_0 is not reached within $2^{|Q|}$ steps. We achieve this by defining B as the gadget in Fig. 3 connected to a modified copy \mathcal{A}' of \mathcal{A} with the same state space. The transitions in \mathcal{A}' are defined as follows, where x is the entry state of the gadget (see Fig. 4): for all $q \in Q$ let (i) $\delta_B(q) = x \wedge \delta_{\mathcal{A}}(q)$ if $q \neq q_0$, and (ii) $\delta_B(q_0) = q_0 \vee (x \wedge \delta_{\mathcal{A}}(q_0))$. Thus, q_0 has a self-loop, and given a set $S \subseteq Q$ in the automaton \mathcal{A} , if $q_0 \notin S$, then $\text{Pre}_{\mathcal{A}}(S) = \text{Pre}_B(S \cup \{x\})$ that is Pre_B mimics $\text{Pre}_{\mathcal{A}}$ when x is in the argument (and q_0 has not been reached yet). Note that if $x \notin S$ (and $q_0 \notin S$), then $\text{Pre}_B(S) = \emptyset$, that is unless q_0 has been reached, the simulation of \mathcal{A} by B stops. Since we need that B mimics \mathcal{A} for $2^{|Q|}$ steps, we define the gadget and the set \mathcal{F}' to ensure that $x \in \mathcal{F}'$ and if $x \in \text{Pre}_B^i(\mathcal{F}')$, then $x \in \text{Pre}_B^{i+1}(\mathcal{F}')$ for all $i \leq 2^{|Q|}$.

In the gadget (Fig. 3), the state x has nondeterministic transitions $\delta_B(x) = c_0^1 \vee c_0^2 \vee \dots \vee c_0^n$ to n components with state space $C_i = \{c_0^i, \dots, c_{p_i-1}^i\}$ where p_i is the $(i+1)$ -th prime number, and the transitions¹ $\delta_B(c_j^i) = x \wedge c_{j+1}^i$ form a loop in each component ($i = 1, \dots, n$). We choose n such that $p_n^\# = \prod_{i=1}^n p_i > 2^{|Q|}$ (take $n = |Q|$). Note that the number of states in the gadget is $1 + \sum_{i=1}^n p_i \in O(n^2 \log n)$ [4] and hence the construction is polynomial in the size of \mathcal{A} .

By construction, for all sets S , we have $x \in \text{Pre}_B(S)$ whenever the first state c_0^i of some component C_i is in S , and if $x \in S$, then $c_j^i \in S$ implies $c_{j-1}^i \in \text{Pre}_B(S)$. Thus, if $x \in S$, the operator $\text{Pre}_B(S)$ ‘shifts’ backward the states in each component; and, x is in the next iteration (i.e., $x \in \text{Pre}_B(S)$) as long as $c_0^i \in S$ for some component C_i .

Now, define the set of accepting states \mathcal{F}' in B in such a way that all states c_0^i disappear simultaneously only after $p_n^\#$ iterations. Let $\mathcal{F}' = \mathcal{F} \cup \{x\} \cup \bigcup_{1 \leq i \leq n} (C_i \setminus \{c_{p_i-1}^i\})$, thus \mathcal{F}' contains all states of the gadget except the last state of each component. It is easy to check that, irrespective of the transition relation in \mathcal{A} , we have $x \in \text{Pre}_B^i(\mathcal{F}')$ if and only if $0 \leq i < p_n^\#$. Therefore, if $q_0 \in \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for some i , then $q_0 \in \text{Pre}_B^j(\mathcal{F}')$ for all $j \geq i$ by the self-loop on q_0 . On the other hand, if $q_0 \notin \text{Pre}_{\mathcal{A}}^i(\mathcal{F})$ for all $i \geq 0$, then since $x \notin \text{Pre}_B^i(\mathcal{F}')$ for all $i > p_n^\#$, we have $\text{Pre}_B^i(\mathcal{F}') = \emptyset$ for all $i > p_n^\#$. This shows that the language of \mathcal{A} is non-empty if and only if the language of B is infinite from some state (namely q_0), and establishes the correctness of the reduction. \square

Relation with MDPs The underlying structure of a Markov decision process $\mathcal{M} = \langle Q, \mathbf{A}, \delta \rangle$ is an alternating graph, where the successor q' of a state q is obtained by an existential choice of an action a and a universal choice of a state $q' \in \text{Supp}(\delta(q, a))$. Therefore, it is natural that some questions related to MDPs have a corresponding formulation in terms of alternating automata. We show that such connections exist between synchronizing problems for MDPs and language-theoretic questions for alternating automata, such as emptiness and universal finiteness. Given a 1L-AFA $\mathcal{A} = \langle Q, \delta_{\mathcal{A}}, \mathcal{F} \rangle$, assume without loss of generality that the transition function $\delta_{\mathcal{A}}$ is such that $\delta_{\mathcal{A}}(q) = c_1 \vee \dots \vee c_m$ has the same number m of conjunctive clauses for all $q \in Q$. From \mathcal{A} , construct the MDP $\mathcal{M}_{\mathcal{A}} = \langle Q, \mathbf{A}, \delta_{\mathcal{M}} \rangle$ where $\mathbf{A} = \{a_1, \dots, a_m\}$ and $\delta_{\mathcal{M}}(q, a_k)$ is the uniform distribution over the states occurring in the k -th clause c_k in $\delta_{\mathcal{A}}(q)$, for all $q \in Q$ and $a_k \in \mathbf{A}$. Then, we have $\text{Acc}_{\mathcal{A}}(n, \mathcal{F}) = \text{Pre}_{\mathcal{M}_{\mathcal{A}}}^n(\mathcal{F})$ for all $n \geq 0$. Similarly, from an MDP \mathcal{M} and a set T of states, we can construct a 1L-AFA $\mathcal{A} = \langle Q, \delta_{\mathcal{A}}, \mathcal{F} \rangle$ with $\mathcal{F} = T$ such that $\text{Acc}_{\mathcal{A}}(n, \mathcal{F}) = \text{Pre}_{\mathcal{M}}^n(T)$ for all $n \geq 0$: let $\delta_{\mathcal{A}}(q) = \bigvee_{a \in \mathbf{A}} \bigwedge_{q' \in \text{post}(q, a)} q'$ for all $q \in Q$. For example, for $\mathbf{A} = \{a, b\}$ if $\delta_{\mathcal{M}}(q_1, a)(q_2) = \delta_{\mathcal{M}}(q_1, a)(q_3) = \frac{1}{2}$ and $\delta_{\mathcal{M}}(q_1, b)(q_4) = 1$, then $\delta_{\mathcal{A}}(q_1) = (q_2 \wedge q_3) \vee q_4$.

¹ In expression c_j^i , we assume that j is interpreted modulo p_i .

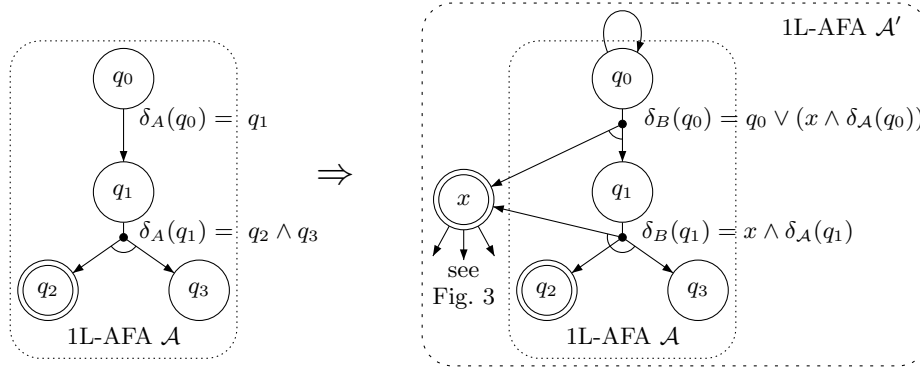


Fig. 4. Detail of the copy \mathcal{A}' obtained from \mathcal{A} in the reduction of Fig. 3.

It follows that 1L-AFA and MDPs have the same structure of alternating graph, and that, up to the correspondence between 1L-AFA and MDPs established above, $Acc_{\mathcal{A}}(n, T) = Pre_{\mathcal{M}}^n(T)$, which justifies to denote $Acc_{\mathcal{A}}(n, T)$ by $Pre_{\mathcal{A}}^n(T)$. Several decision problems for 1L-AFA can be solved by computing the sequence $Acc_{\mathcal{A}}(n, \mathcal{F})$ (i.e., $Pre_{\mathcal{A}}^n(\mathcal{F})$), and analogously we show that synchronizing problems for MDPs can also be solved by computing the sequence $Pre_{\mathcal{M}}^n(\mathcal{F})$. Therefore, the above relationship between 1L-AFA and MDPs provides a tight connection that we use in Section 3 to transfer complexity results between 1L-AFA and MDPs.

3 Eventually Synchronizing

In this section, we show the PSPACE-completeness of the membership problem for eventually synchronizing objectives and the three winning modes. By Lemma 3 and the remark at the end of Section 2.2, we consider the membership problem with function *sum* and Dirac initial distributions (i.e., single initial state).

The eventually synchronizing objective is reminiscent of a reachability objective in the distribution-based semantics: it requires that in the sequence of distributions of an MDP \mathcal{M} under strategy α we have $\sup_n \mathcal{M}_n^\alpha(T) = 1$ (and that the sup is reached in the case of sure winning, that is $\mathcal{M}_n^\alpha(T) = 1$ for some $n \geq 0$).

The sure winning mode can be solved by a reachability analysis in the alternating graph underlying the MDP (Section 3.1). We show that the almost-sure winning mode can be solved by a reduction to the limit-sure winning mode (Section 3.2). We solve the limit-sure winning mode by a reduction to a reachability question in a modified MDP of exponential size that ensures the probability mass reaches the target set synchronously (Section 3.3). We present reductions to show PSPACE-hardness of each winning mode, matching our PSPACE upper bounds.

3.1 Sure eventually synchronizing

Given a target set T , the membership problem for sure-winning eventually synchronizing objective in T can be solved by computing the sequence $Pre^n(T)$ of iterated predecessors,

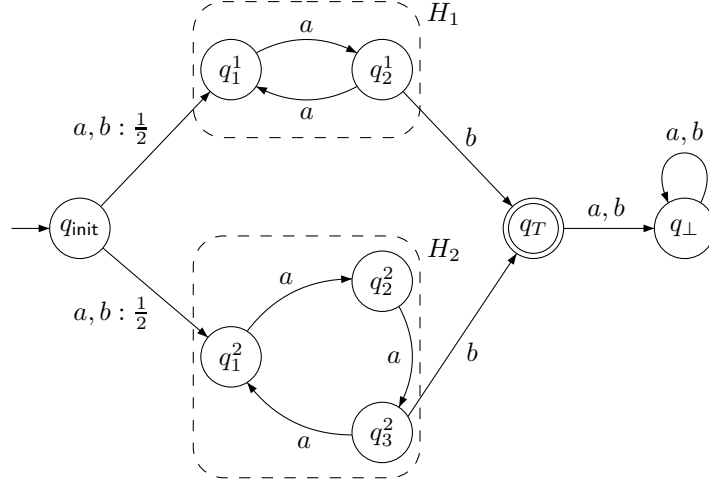


Fig. 5. The MDP \mathcal{M}_2 .

like in 1L-AFA. A state q_{init} is sure-winning for eventually synchronizing in T if $q_{\text{init}} \in \text{Pre}^n(T)$ for some $n \geq 0$.

Lemma 5. *Let \mathcal{M} be an MDP and T be a target set. For all states q_{init} , we have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T)$ if and only if there exists $n \geq 0$ such that $q_{\text{init}} \in \text{Pre}_{\mathcal{M}}^n(T)$.*

Proof. We prove the following equivalence by induction (on the length i): for all initial states q_{init} , there exists a strategy α sure-winning in i steps from q_{init} (i.e., such that $\mathcal{M}_i^\alpha(T) = 1$) if and only if $q_{\text{init}} \in \text{Pre}^i(T)$. The case $i = 0$ trivially holds since for all strategies α , we have $\mathcal{M}_0^\alpha(T) = 1$ if and only if $q_{\text{init}} \in T$.

Assume that the equivalence holds for all $i < n$. For the induction step, show that \mathcal{M} is sure eventually synchronizing from q_{init} (in n steps) if and only if there exists an action a such that \mathcal{M} is sure eventually synchronizing (in $n - 1$ steps) from all states $q' \in \text{post}(q_{\text{init}}, a)$ (equivalently, $\text{post}(q_{\text{init}}, a) \subseteq \text{Pre}^{n-1}(T)$ by the induction hypothesis, that is $q_{\text{init}} \in \text{Pre}^n(T)$). First, if all successors q' of q_{init} under some action a are sure eventually synchronizing, then so is q_{init} by playing a followed by a winning strategy from each successor q' . For the other direction, assume towards contradiction that \mathcal{M} is sure eventually synchronizing from q_{init} (in n steps), but for each action a , there is a state $q' \in \text{post}(q_{\text{init}}, a)$ that is not sure eventually synchronizing. Then, from q' there is a positive probability to reach a state not in T after $n - 1$ steps, no matter the strategy played. Hence from q_{init} , for all strategies, the probability mass in T cannot be 1 after n steps, in contradiction with the fact that \mathcal{M} is sure eventually synchronizing from q_{init} in n steps. It follows that the induction step holds, and the proof is complete. \square

By Lemma 5, the membership problem for sure eventually synchronizing is equivalent to the emptiness problem of 1L-AFA, and thus PSPACE-complete. Moreover if $q_{\text{init}} \in \text{Pre}_{\mathcal{M}}^n(T)$, a finite-memory strategy with n modes that at mode i in a state q plays an action a such that $\text{post}(q, a) \subseteq \text{Pre}^{i-1}(T)$ is sure winning for eventually synchronizing.

There exists a family of MDPs \mathcal{M}_n ($n \in \mathbb{N}$) over alphabet $\{a, b\}$ that are sure winning for eventually synchronizing, and where the sure winning strategies require exponential

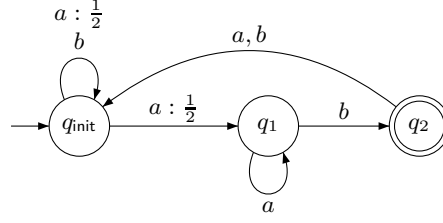


Fig. 6. An MDP where infinite memory is necessary for almost-sure eventually and almost-sure weakly synchronizing strategies.

memory. The MDP \mathcal{M}_2 is shown in Fig. 5. The structure of \mathcal{M}_n is an initial uniform probabilistic transition to n components H_1, \dots, H_n where H_i is a cycle of length p_i the i th prime number. On action a , the next state in the cycle is reached, and on action b the target state q_T is reached, only from the last state in the cycles. From other states, the action b leads to q_\perp (transitions not depicted). A sure winning strategy for eventually synchronizing in $\{q_T\}$ is to play a in the first $p_n^\# = \prod_{i=1}^n p_i$ steps, and then play b . This requires memory of size $p_n^\# > 2^n$ while the size of \mathcal{M}_n is in $O(n^2 \log n)$ [4]. It can be proved by standard pumping arguments that no strategy of size smaller than $p_n^\#$ is sure winning.

The following theorem summarizes the results for sure eventually synchronizing.

Theorem 2. *For sure eventually synchronizing in MDPs:*

1. (Complexity). *The membership problem is PSPACE-complete.*
2. (Memory). *Exponential memory is necessary and sufficient for both pure and randomized strategies, and pure strategies are sufficient.*

3.2 Almost-sure eventually synchronizing

We show an example where infinite memory is necessary to win for almost-sure eventually synchronizing. Consider the MDP in Fig. 6 with initial state q_{init} . We construct a strategy that is almost-sure eventually synchronizing in q_2 , showing that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(q_2)$. First, observe that for all $\varepsilon > 0$ we can have probability at least $1 - \varepsilon$ in q_2 after finitely many steps: playing n times a and then b leads to probability $1 - \frac{1}{2^n}$ in q_2 . Choosing n sufficiently large (namely, $n > \log_2(\frac{1}{\varepsilon})$) shows that the MDP is limit-sure eventually synchronizing in q_2 . Moreover the remaining probability mass is in q_{init} . It turns out that from any (initial) distribution with support $\{q_{\text{init}}, q_2\}$, the MDP is again limit-sure eventually synchronizing in q_2 , and with support in $\{q_{\text{init}}, q_2\}$. Therefore we can take a smaller value of ε and play a strategy to have probability at least $1 - \varepsilon$ in q_2 , and repeat this for $\varepsilon \rightarrow 0$. This strategy ensures almost-sure eventually synchronizing in q_2 . The next result shows that infinite memory is necessary for almost-sure winning in this example.

Lemma 6. *There exists an almost-sure eventually synchronizing MDP for which all almost-sure eventually synchronizing strategies require infinite memory.*

Proof. Consider the MDP \mathcal{M} shown in Fig. 6. We argued before the lemma that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(q_2)$ and we now show that infinite memory is necessary from q_{init} for almost-sure eventually synchronizing in q_2 . Note that \mathcal{M} is not sure eventually synchronizing in q_2 since the probability in q_{init} is positive at all times (for all strategies).

Assume towards contradiction that there exists a finite-memory strategy α that is almost-sure eventually synchronizing in q_2 . Consider the Markov chain $\mathcal{M}(\alpha)$ (the product of the MDP \mathcal{M} with the finite-state transducer defining α). A state (q, m) in $\mathcal{M}(\alpha)$ is called a q -state. Since α is almost-sure eventually synchronizing (but is not sure eventually synchronizing) in q_2 , there is a q_2 -state in the recurrent states of $\mathcal{M}(\alpha)$. Since on all actions q_{init} is a successor of q_2 , and q_{init} is a successor of itself, it follows that there is a recurrent q_{init} -state in $\mathcal{M}(\alpha)$, and that all periodic classes of recurrent states in $\mathcal{M}(\alpha)$ contain a q_{init} -state. Hence, in each stationary distribution there is a q_{init} -state with a positive probability, and therefore the probability mass in q_{init} is bounded away from zero. It follows that the probability mass in q_2 is bounded away from 1 thus α is not almost-sure eventually synchronizing in q_2 , a contradiction. \square

The membership problem for almost-sure eventually synchronizing can be reduced to other winning modes since an almost-sure eventually synchronizing strategy is either sure eventually synchronizing or almost-sure weakly synchronizing. Nevertheless we give a direct proof that the problem is decidable in PSPACE, using a characterization that will be useful later for almost-sure weakly synchronizing.

It turns out that in general, almost-sure eventually synchronizing strategies can be constructed from a family of limit-sure eventually synchronizing strategies if we can also ensure that the probability mass remains in the winning region (as in the MDP in Fig. 6). We present a characterization of the winning region for almost-sure winning based on an extension of the limit-sure eventually synchronizing objective *with exact support*. This objective requires to ensure probability arbitrarily close to 1 in the target set T , and moreover that after the same number of steps the support of the probability distribution is contained in the given set U . Formally, given an MDP \mathcal{M} , let $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$ for $T \subseteq U$ be the set of all initial distributions such that for all $\varepsilon > 0$ there exists a strategy α and $n \in \mathbb{N}$ such that $\mathcal{M}_n^\alpha(T) \geq 1 - \varepsilon$ and $\mathcal{M}_n^\alpha(U) = 1$. We say that α is limit-sure eventually synchronizing in T with support in U .

We will present an algorithmic solution to limit-sure eventually synchronizing objectives with exact support in Section 3.3. Our characterization of the winning region for almost-sure winning is as follows.

Lemma 7. *Let \mathcal{M} be an MDP and T be a target set. For all states q_{init} , we have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_T)$ if and only if there exists a set U of states such that:*

- $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$, and
- $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$ where d_U is the uniform distribution over U .

Proof. First, if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_T)$, then there is a strategy α such that $\sup_{n \in \mathbb{N}} \mathcal{M}_n^\alpha(T) = 1$. Then either $\mathcal{M}_n^\alpha(T) = 1$ for some $n \geq 0$, or $\limsup_{n \rightarrow \infty} \mathcal{M}_n^\alpha(T) = 1$. If $\mathcal{M}_n^\alpha(T) = 1$, then q_{init} is sure winning for eventually synchronizing in T , thus $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T)$ and we can take $U = T$. Otherwise, for all $i > 0$ there exists $n_i \in \mathbb{N}$ such that $\mathcal{M}_{n_i}^\alpha(T) \geq 1 - 2^{-i}$, and moreover $n_{i+1} > n_i$ for all $i > 0$. Let $s_i = \text{Supp}(\mathcal{M}_{n_i}^\alpha)$ be the support of $\mathcal{M}_{n_i}^\alpha$. Since the state space is finite, there is a set U that occurs infinitely often in the sequence $s_0 s_1 \dots$, thus for all $k > 0$ there exists $m_k \in \mathbb{N}$ such that $\mathcal{M}_{m_k}^\alpha(T) \geq 1 - 2^{-k}$ and $\mathcal{M}_{m_k}^\alpha(U) = 1$. It follows that α is sure eventually synchronizing in U from q_{init} , hence $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$. Moreover \mathcal{M} with initial distribution $d_1 = \mathcal{M}_{m_1}^\alpha$ is limit-sure eventually synchronizing in T with exact support in U . Since $\text{Supp}(d_1) = U = \text{Supp}(d_U)$, it follows by Corollary 2 that $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$.

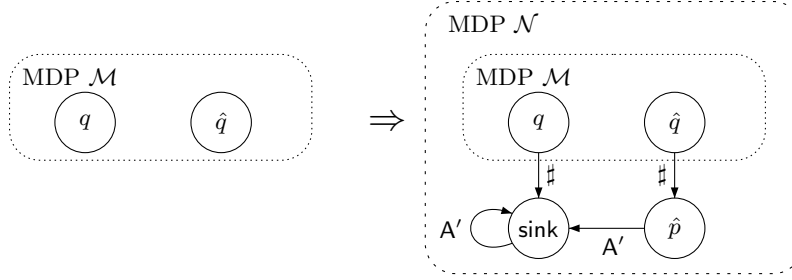


Fig. 7. Sketch of the reduction to show PSPACE-hardness of the membership problem for almost-sure eventually synchronizing.

To establish the converse, note that since $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$, it follows from Corollary 2 that from all initial distributions with support in U , for all $\varepsilon > 0$ there exists a strategy α_ε and a position n_ε such that $\mathcal{M}_{n_\varepsilon}^{\alpha_\varepsilon}(T) \geq 1 - \varepsilon$ and $\mathcal{M}_{n_\varepsilon}^{\alpha_\varepsilon}(U) = 1$. We construct an almost-sure limit eventually synchronizing strategy α as follows. Since $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$, play according to a sure eventually synchronizing strategy from q_{init} until all the probability mass is in U . Then for $i = 1, 2, \dots$ and $\varepsilon_i = 2^{-i}$, repeat the following procedure: given the current probability distribution, select the corresponding strategy α_{ε_i} and play according to α_{ε_i} for n_{ε_i} steps, ensuring probability mass at least $1 - 2^{-i}$ in T , and since after that the support of the probability mass is again in U , play according to $\alpha_{\varepsilon_{i+1}}$ for $n_{\varepsilon_{i+1}}$ steps, etc. This strategy α ensures that $\sup_{n \in \mathbb{N}} \mathcal{M}_n^\alpha(T) = 1$ from q_{init} , hence $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_T)$. \square

Note that from Lemma 7, it follows that counting strategies are sufficient to win almost-sure eventually synchronizing objective (a strategy is *counting* if $\alpha(\rho) = \alpha(\rho')$ for all prefixes ρ, ρ' with the same length and $\text{Last}(\rho) = \text{Last}(\rho')$).

As we show in Section 3.3 that the membership problem for limit-sure eventually synchronizing with exact support can be solved in PSPACE, it follows from the characterization in Lemma 7 that the membership problem for almost-sure eventually synchronizing is in PSPACE, using the following (N)PSPACE algorithm: guess the set U , and check that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$, and that $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$ where d_U is the uniform distribution over U (this can be done in PSPACE by Theorem 2 and Theorem 4). We present a matching lower bound.

Lemma 8. *The membership problem for $\langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_T)$ is PSPACE-hard even if T is a singleton.*

Proof. We show the result by a reduction from the membership problem for sure eventually synchronizing, which is PSPACE-complete by Theorem 2. Given an MDP $\mathcal{M} = \langle Q, A, \delta \rangle$, an initial state $q_{\text{init}} \in Q$, and a state $\hat{q} \in Q$, we construct an MDP $\mathcal{N} = \langle Q', A', \delta' \rangle$ with $Q \subseteq Q'$ and a state $\hat{p} \in Q'$ such that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\hat{q})$ in \mathcal{M} if and only if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\hat{p})$ in \mathcal{N} . The MDP \mathcal{N} is a copy of \mathcal{M} with two new states \hat{p} and sink reachable only by a new action $\#$ (see Fig. 7). Formally, $Q' = Q \cup \{\hat{p}, \text{sink}\}$ and $A' = A \cup \{\#\}$, and the transition function δ' is defined as follows, for all $q \in Q$ and $a \in A$: $\delta'(q, a) = \delta(q, a)$, and $\delta'(q, \#)(\text{sink}) = 1$ if $q \neq \hat{q}$, and $\delta'(\hat{q}, \#)(\hat{p}) = 1$; finally, for all $a \in A'$, let $\delta'(\hat{p}, a)(\text{sink}) = \delta'(\text{sink}, a)(\text{sink}) = 1$.

The goal is that \mathcal{N} simulates \mathcal{M} until the action \sharp is played in \hat{q} to move the probability mass from \hat{q} to \hat{p} , ensuring that if \mathcal{M} is sure-winning for eventually synchronizing in \hat{q} , then \mathcal{N} is also sure-winning (and thus almost-sure winning) for eventually synchronizing in \hat{p} . Moreover, the only way to be almost-sure eventually synchronizing in \hat{p} is to have probability 1 in \hat{p} at some point, because the state \hat{p} is transient under all strategies, thus the probability mass cannot accumulate and tend to 1 in \hat{p} in the long run. Therefore (from all initial states q_{init}) \mathcal{M} is sure-winning for eventually synchronizing in \hat{q} if and only if \mathcal{N} is almost-sure winning for eventually synchronizing in \hat{p} . It follows from this reduction that the membership problem for almost-sure eventually synchronizing objective is PSPACE-hard. \square

The results of this section are summarized as follows.

Theorem 3. *For almost-sure eventually synchronizing in MDPs:*

1. (Complexity). *The membership problem is PSPACE-complete.*
2. (Memory). *Infinite memory is necessary in general for both pure and randomized strategies, and pure strategies are sufficient.*

3.3 Limit-sure eventually synchronizing

In this section, we present the algorithmic solution for limit-sure eventually synchronizing with exact support, which requires to get probability arbitrarily close to 1 in a target set T while all the probability mass is contained in a given set U . Note that the limit-sure eventually synchronizing objective is a special case where the support is the state space of the MDP. Consider the MDP in Fig. 1 which is limit-sure eventually synchronizing in $\{q_2\}$, as shown in Lemma 2. For $i = 0, 1, \dots$, the sequence $\text{Pre}^i(T)$ of predecessors of $T = \{q_2\}$ is ultimately periodic: $\text{Pre}^0(T) = \{q_2\}$, and $\text{Pre}^i(T) = \{q_1\}$ for all $i \geq 1$. Given $\varepsilon > 0$, a strategy to get probability $1 - \varepsilon$ in q_2 first accumulates probability mass in the *periodic* subsequence of predecessors (here $\{q_1\}$), and when the probability mass is greater than $1 - \varepsilon$ in q_1 , the strategy injects the probability mass in q_2 (through the aperiodic prefix of the sequence of predecessors). This is the typical shape of a limit-sure eventually synchronizing strategy. Note that in this scenario, the MDP is also limit-sure eventually synchronizing in every set $\text{Pre}^i(T)$ of the sequence of predecessors. A special case is when it is possible to get probability 1 in the sequence of predecessors after finitely many steps. In this case, the probability mass injected in T is 1 and the MDP is even sure-winning. The algorithm for deciding limit-sure eventually synchronizing relies on the above characterization, generalized in Lemma 9 to limit-sure eventually synchronizing with exact support, saying that limit-sure eventually synchronizing in T with support in U is equivalent to either sure eventually synchronizing in T (and therefore also in U), or limit-sure eventually synchronizing in $\text{Pre}^k(T)$ with support in $\text{Pre}^k(U)$ (for arbitrary k). The intuition of the proof is that if an MDP is limit-sure eventually synchronizing in T with support in U , then either a bounded number of steps is sufficient to get probability $1 - \varepsilon$ in T (and then we argue that the MDP is sure eventually synchronizing), or unbounded number of steps is required, which means that k steps before getting probability $1 - \varepsilon$ in T , the probability mass in $\text{Pre}^k(T)$ must also be close to 1 (and arbitrarily close to 1 as ε tends to 0).

Lemma 9. *For all $T \subseteq U$ and all $k \geq 0$, we have $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U) = \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T) \cup \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z)$ where $R = \text{Pre}^k(T)$ and $Z = \text{Pre}^k(U)$.*

Proof. We proceed in two parts. First we show that $\langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T) \cup \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z) \subseteq \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$; since $T \subseteq U$, it follows from the definitions that $\langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T) \subseteq \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$; to show that $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z) \subseteq \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$ in an MDP \mathcal{M} , let $\varepsilon > 0$ and consider an initial distribution d_0 and a strategy α such that for some $i \geq 0$ we have $\mathcal{M}_i^\alpha(R) \geq 1 - \varepsilon$ and $\mathcal{M}_i^\alpha(Z) = 1$. We construct a strategy β that plays like α for the first i steps, and then since $R = \text{Pre}^k(T)$ and $Z = \text{Pre}^k(U)$ plays from states in R according to a sure eventually synchronizing strategy with target T , and from states in $Z \setminus R$ according to a sure eventually synchronizing strategy with target U (such strategies exist by the proof of Lemma 5). The strategy β ensures from d_0 that $\mathcal{M}_{i+k}^\beta(T) \geq 1 - \varepsilon$ and $\mathcal{M}_{i+k}^\beta(U) = 1$, showing that \mathcal{M} is limit-sure eventually synchronizing in T with support in U .

Second we show the converse inclusion, namely that $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U) \subseteq \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T) \cup \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z)$. Consider an initial distribution $d_0 \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U)$ in the MDP \mathcal{M} and for $\varepsilon_i = \frac{1}{i}$ ($i \in \mathbb{N}$) let α_i be a strategy and $n_i \in \mathbb{N}$ such that $\mathcal{M}_{n_i}^{\alpha_i}(T) \geq 1 - \varepsilon_i$ and $\mathcal{M}_{n_i}^{\alpha_i}(U) = 1$. We consider two cases. (a) If the set $\{n_i \mid i \geq 0\}$ is bounded, then there exists a number n that occurs infinitely often in the sequence $(n_i)_{i \in \mathbb{N}}$, and such that for all $i \geq 0$, there exists a strategy β_i such that $\mathcal{M}_n^{\beta_i}(T) \geq 1 - \varepsilon$ and $\mathcal{M}_n^{\beta_i}(U) = 1$. Since n is fixed, we can assume w.l.o.g. that the strategies β_i are pure, and since there is a finite number of pure strategies over paths of length at most n , it follows that there is a strategy β that occurs infinitely often among the strategies β_i and such that for all $\varepsilon > 0$ we have $\mathcal{M}_n^\beta(T) \geq 1 - \varepsilon$, hence $\mathcal{M}_n^\beta(T) = 1$, showing that \mathcal{M} is sure winning for eventually synchronizing in T , that is $d_0 \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T)$. (b) otherwise, the set $\{n_i \mid i \geq 0\}$ is unbounded and we can assume w.l.o.g. that $n_i \geq k$ for all $i \geq 0$. We claim that the family of strategies α_i ensures limit-sure eventually synchronizing in $R = \text{Pre}^k(T)$ with support in $Z = \text{Pre}^k(U)$. Essentially this is because if the probability in T is close to 1 after n_i steps, then k steps before the probability in $\text{Pre}^k(T)$ must be close to 1 as well. Formally, we show that α_i is such that $\mathcal{M}_{n_i-k}^{\alpha_i}(R) \geq 1 - \frac{\varepsilon}{\eta^k}$ and $\mathcal{M}_{n_i-k}^{\alpha_i}(Z) = 1$ where η is the smallest positive probability in the transitions of \mathcal{M} . Towards contradiction, assume that $\mathcal{M}_{n_i-k}^{\alpha_i}(R) < 1 - \frac{\varepsilon}{\eta^k}$, then $\mathcal{M}_{n_i-k}^{\alpha_i}(Q \setminus R) > \frac{\varepsilon}{\eta^k}$ and from every state $q \in Q \setminus R$, no matter which sequence of actions is played by α_i for the next k steps, there is a path from q to a state outside of T , thus with probability at least η^k . Hence the probability in $Q \setminus T$ after n_i steps is greater than $\frac{\varepsilon}{\eta^k} \cdot \eta^k$, and therefore $\mathcal{M}_{n_i}^{\alpha_i}(T) < 1 - \varepsilon$, in contradiction with the definition of α_i . This shows that $\mathcal{M}_{n_i-k}^{\alpha_i}(R) \geq 1 - \frac{\varepsilon}{\eta^k}$, and an argument analogous to the proof of Lemma 5 shows that $\mathcal{M}_{n_i-k}^{\alpha_i}(Z) = 1$. It follows that $d_0 \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z)$ and the proof is complete. \square

Thanks to Lemma 9, since sure-winning is already solved in Section 3.1, it suffices to solve the limit-sure eventually synchronizing problem for target $R = \text{Pre}^k(T)$ and support $Z = \text{Pre}^k(U)$ with arbitrary k , instead of T and U . We choose k such that both $\text{Pre}^k(T)$ and $\text{Pre}^k(U)$ lie in the periodic part of the sequence of pairs of predecessors $(\text{Pre}^i(T), \text{Pre}^i(U))$. Note that $\text{Pre}^i(T) \subseteq \text{Pre}^i(U) \subseteq Q$ for all $i \geq 0$, and there are at most $3^{|Q|}$ different pairs (A, B) with $A \subseteq B \subseteq Q$ (each state $q \in Q$ belongs either to A , or to $B \setminus A$, or to $Q \setminus B$). Hence we can assume that $k \leq 3^{|Q|}$. For such value of k the limit-sure problem is conceptually simpler: once some probability is injected in $R = \text{Pre}^k(T)$, it can loop through the sequence of predecessors and visit R infinitely often (every r steps, where $r \leq 3^{|Q|}$ is the period of the sequence of pairs of predecessors). It follows that if a strategy ensures with probability 1 that the set R can be reached by finite paths whose lengths are congruent

modulo r , then the whole probability mass can indeed synchronously accumulate in R in the limit.

Therefore, limit-sure eventually synchronizing in R reduces to standard limit-sure reachability (in the state-based semantics) with target set R and the additional requirement that the numbers of steps at which the target set is reached be congruent modulo r . In the case of limit-sure eventually synchronizing with support in Z , we also need to ensure that no mass of probability leaves the sequence $\text{Pre}^i(Z)$. In a state $q \in \text{Pre}^i(Z)$, we say that an action $a \in \mathbf{A}$ is *Z-safe* at position i if² $\text{post}(q, a) \subseteq \text{Pre}^{i-1}(Z)$. In states $q \notin \text{Pre}^i(Z)$ there is no *Z-safe* action at position i .

To encode the above requirements, we construct an MDP $\mathcal{M}_Z \times [r]$ that allows only *Z-safe* actions to be played (and then mimics the original MDP), and tracks the position (modulo r) in the sequence of predecessors, thus simply decrementing the position on each transition since all successors of a state $q \in \text{Pre}^i(Z)$ on a safe action are in $\text{Pre}^{i-1}(Z)$.

Formally, if $\mathcal{M} = \langle Q, \mathbf{A}, \delta \rangle$ then $\mathcal{M}_Z \times [r] = \langle Q', \mathbf{A}, \delta' \rangle$ where

- $Q' = Q \times \{r-1, \dots, 1, 0\} \cup \{\text{sink}\}$; a state $\langle q, i \rangle$ consisting of a state q of \mathcal{M} and a position i in the predecessor sequence corresponds to the promise that $q \in \text{Pre}^i(Z)$;
- δ' is defined as follows (assuming an arithmetic modulo r on positions) for all $\langle q, i \rangle \in Q'$ and $a \in \mathbf{A}$: if a is a *Z-safe* action in q at position i , then $\delta'(\langle q, i \rangle, a)(\langle q', i-1 \rangle) = \delta(q, a)(q')$, otherwise $\delta'(\langle q, i \rangle, a)(\text{sink}) = 1$ (and sink is absorbing).

Note that the size of the MDP $\mathcal{M}_Z \times [r]$ is exponential in the size of \mathcal{M} (since r is at most $3^{|Q|}$).

Lemma 10. *Let \mathcal{M} be an MDP and $R \subseteq Z$ be two sets of states such that $\text{Pre}^r(R) = R$ and $\text{Pre}^r(Z) = Z$ where $r > 0$. Then a state q_{init} is limit-sure eventually synchronizing in R with support in Z ($q_{\text{init}} \in \langle \langle 1 \rangle \rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z)$) if and only if there exists $0 \leq t < r$ such that $\langle q_{\text{init}}, t \rangle$ is limit-sure winning for the reachability objective $\Diamond(R \times \{0\})$ in the MDP $\mathcal{M}_Z \times [r]$.*

Proof. For the first direction of the lemma, assume that q_{init} is limit-sure eventually synchronizing in R with support in Z , and for $\varepsilon > 0$ let β be a strategy such that $\mathcal{M}_k^\beta(Z) = 1$ and $\mathcal{M}_k^\beta(R) \geq 1 - \varepsilon$ for some number k of steps. Let $0 \leq t \leq r$ such that $t = k \bmod r$. Let $R_0 = R \times \{0\}$. We show that from initial state (q_{init}, t) the strategy α in $\mathcal{M}_Z \times [r]$ that mimics (copies) the strategy β is limit-sure winning for the reachability objective $\Diamond R_0$: it follows from Lemma 5 that α plays only *Z-safe* actions, and since $\text{Pr}^\alpha(\Diamond R_0) \geq \text{Pr}^\alpha(\Diamond^k R_0) = \mathcal{M}_k^\beta(R) \geq 1 - \varepsilon$, the result follows.

For the converse direction, assuming that there exists $0 \leq t < r$ such that $\langle q_{\text{init}}, t \rangle$ is limit-sure winning for the reachability objective $\Diamond R_0$ in $\mathcal{M}_Z \times [r]$, we show that q_{init} is limit-sure synchronizing in target set R with exact support in Z . Since the winning region of limit-sure and almost-sure reachability coincide for MDPs [18], there exists a (pure) strategy α in $\mathcal{M}_Z \times [r]$ with initial state $\langle q_{\text{init}}, t \rangle$ such that $\text{Pr}^\alpha(\Diamond R_0) = 1$.

Given $\varepsilon > 0$, we construct from α a pure strategy β in \mathcal{M} that is $(1 - \varepsilon)$ -synchronizing in R with support in Z . Given a finite path $\rho = q_0 a_0 q_1 a_1 \dots q_n$ in \mathcal{M} (with $q_0 = q_{\text{init}}$), there is a corresponding path $\rho' = \langle q_0, k_0 \rangle a_0 \langle q_1, k_1 \rangle a_1 \dots \langle q_n, k_n \rangle$ in $\mathcal{M}_Z \times [r]$ where $k_0 = t$ and $k_{i+1} = k_i - 1$ for all $i \geq 0$. Since the sequence k_0, k_1, \dots is uniquely determined from ρ , there is a clear bijection between the paths in \mathcal{M} starting in q_{init} and the paths

² Since $\text{Pre}^r(Z) = Z$ and $\text{Pre}^r(R) = R$, we assume a modular arithmetic for exponents of Pre , that is $\text{Pre}^x(\cdot)$ is defined as $\text{Pre}^{x \bmod r}(\cdot)$. For example $\text{Pre}^{-1}(Z)$ is $\text{Pre}^{r-1}(Z)$.

in $\mathcal{M}_Z \times [r]$ starting in $\langle q_{\text{init}}, t \rangle$. In the sequel, we freely omit to apply and mention this bijection. Define the strategy β as follows: if $q_n \in \text{Pre}^{k_n}(R)$, then there exists an action a such that $\text{post}(q_n, a) \subseteq \text{Pre}^{k_n-1}(R)$ and we define $\beta(\rho) = a$, otherwise let $\beta(\rho) = \alpha(\rho')$. Thus β mimics α (thus playing only Z -safe actions) unless a state q is reached at step n such that $q \in \text{Pre}^{t-n}(R)$, and then β switches to always playing actions that are R -safe (and thus also Z -safe since $R \subseteq Z$). We now prove that β is limit-sure eventually synchronizing in target set R with support in Z . First since β plays only Z -safe actions, it follows for all k such that $t - k = 0$ (modulo r), all states reached from q_{init} with positive probability after k steps are in Z . Hence $\mathcal{M}_k^\beta(Z) = 1$ for all such k . Second, we show that given $\varepsilon > 0$ there exists k such that $t - k = 0$ and $\mathcal{M}_k^\beta(R) \geq 1 - \varepsilon$, thus also $\mathcal{M}_k^\beta(Z) = 1$ and β is limit-sure eventually synchronizing in target set R with support in Z . To show this, recall that $\Pr^\alpha(\Diamond R_0) = 1$, and therefore $\Pr^\alpha(\Diamond^{\leq k} R_0) \geq 1 - \varepsilon$ for all sufficiently large k . Without loss of generality, consider such a k satisfying $t - k = 0$ (modulo r). For $i = 1, \dots, r - 1$, let $R_i = \text{Pre}^i(R) \times \{i\}$. Then trivially $\Pr^\alpha(\Diamond^{\leq k} \bigcup_{i=0}^r R_i) \geq 1 - \varepsilon$ and since β agrees with α on all finite paths that do not (yet) visit $\bigcup_{i=0}^r R_i$, given a path ρ that visits $\bigcup_{i=0}^r R_i$ (for the first time), only R -safe actions will be played by β and thus all continuations of ρ in the outcome of β will visit R after k steps (in total). It follows that $\Pr^\beta(\Diamond^k R_0) \geq 1 - \varepsilon$, that is $\mathcal{M}_k^\beta(R) \geq 1 - \varepsilon$. Note that we used the same strategy β for all $\varepsilon > 0$ and thus β is also almost-sure eventually synchronizing in R . \square

From the proof of Lemma 10 (last sentence), it follows that if the MDP \mathcal{M} is limit-sure eventually synchronizing in R with support in Z , then \mathcal{M} is also almost-sure eventually synchronizing in R . Since almost-sure synchronization implies limit-sure synchronization by definition, the two notions coincide in this case.

Corollary 1. *Given $R \subseteq Z$ two sets of states in an MDP such that $\text{Pre}^r(R) = R$ and $\text{Pre}^r(Z) = Z$ where $r > 0$, we have $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z) = \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_R)$.*

Since deciding limit-sure reachability is PTIME-complete, it follows from Lemma 10 that limit-sure eventually synchronizing (with exact support) can be decided in EXPTIME. We show in Lemma 11 that the problem can be solved in PSPACE by exploiting the special structure of the exponential MDP used in Lemma 10. We conclude this section by Lemma 12 showing that limit-sure eventually synchronizing with exact support is PSPACE-complete (even in the special case where the support is the whole state space).

Lemma 11. *The membership problem for limit-sure eventually synchronizing with exact support is in PSPACE.*

Proof. We present a (nondeterministic) PSPACE algorithm to decide, given an MDP $\mathcal{M} = \langle Q, A, \delta \rangle$, a state q_{init} , and two sets $T \subseteq U$, whether q_{init} is limit-sure eventually synchronizing in T with support in U .

First, the algorithm computes numbers $k \geq 0$ and $r > 0$ such that for $R = \text{Pre}^k(T)$ and $Z = \text{Pre}^k(U)$ we have $\text{Pre}^r(R) = R$ and $\text{Pre}^r(Z) = Z$. As discussed before, this can be done by guessing $k, r \leq 3^{|Q|}$. By Lemma 9, we have $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, U) = \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_R, Z) \cup \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T)$, and since sure eventually synchronizing in T can be decided in PSPACE (by Theorem 2), it suffices to decide limit-sure eventually synchronizing in R with support in Z in PSPACE. According to Lemma 10, it is therefore sufficient to show that deciding limit-sure winning for the (standard) reachability objective $\Diamond(R \times \{0\})$ in the MDP $\mathcal{M}_Z \times [r]$ can be done in polynomial space. As we cannot afford to construct the exponential-size

MDP $\mathcal{M}_Z \times [r]$, the algorithm relies on the following characterization of the limit-sure winning set for reachability objectives in MDPs. It is known that the winning region for limit-sure and almost-sure reachability coincide [18], and pure memoryless strategies are sufficient. Therefore, we can see that the almost-sure winning set W for the reachability objective $\Diamond(R \times \{0\})$ satisfies the following property: there exists a memoryless strategy $\alpha : W \rightarrow A$ such that (1) W is closed, that is $\text{post}(q, \alpha(q)) \subseteq W$ for all $q \in W$, and (2) in the graph of the Markov chain $M(\alpha)$, for every state $q \in W$, there is a path (of length at most $|W|$) from q to $R \times \{0\}$.

This property ensures that from every state in W , the target set $R \times \{0\}$ is reached within $|W|$ steps with positive (and bounded) probability, and since W is closed it ensures that $R \times \{0\}$ is reached with probability 1 in the long run. Thus any set W satisfying the above property is almost-sure winning.

Our algorithm will guess and explore on the fly a set W to ensure that it satisfies this property, and contains the state $\langle q_{\text{init}}, t \rangle$ for some $t < r$. As we cannot afford to explicitly guess W (remember that W could be of exponential size), we decompose W into *slices* W_0, W_1, \dots such that $W_i \subseteq Q$ and $W_i \times \{-i \bmod r\} = W \cap (Q \times \{-i \bmod r\})$. We start by guessing W_0 , and we use the property that in $\mathcal{M}_Z \times [r]$, from a state (q, j) under all Z -safe actions, all successors are of the form $(\cdot, j-1)$. It follows that the successors of the states in $W_i \times \{-i\}$ should lie in the slice $W_{i+1} \times \{-i-1\}$, and we can guess on the fly the next slice $W_{i+1} \subseteq Q$ by guessing for each state q in a slice W_i an action a_q such that $\bigcup_{q \in W_i} \text{post}(q, a_q) \subseteq W_{i+1}$. Moreover, we need to check the existence of a path from every state in W to $R \times \{0\}$. As W is closed, it is sufficient to check that there is a path from every state in $W_0 \times \{0\}$ to $R \times \{0\}$. To do this we guess along with the slices W_0, W_1, \dots a sequence of sets P_0, P_1, \dots where $P_i \subseteq W_i$ contains the states of slice W_i that belong to the guessed paths. Formally, $P_0 = W_0$, and for all $i \geq 0$, the set P_{i+1} is such that $\text{post}(q, a_q) \cap P_{i+1} \neq \emptyset$ for all $q \in P'_i$ (where $P'_i = P_i \setminus R$ if i is a multiple of r , and $P'_i = P_i$ otherwise), that is P_{i+1} contains a successor of every state in P_i that is not already in the target R (at position 0 modulo r).

We need polynomial space to store the first slice W_0 , the current slice W_i and the set P_i , and the value of i (in binary). As $\mathcal{M}_Z \times [r]$ has $|Q| \cdot r$ states, the algorithm runs for $|Q| \cdot r$ iterations and then checks that (1) $W_{|Q| \cdot r} \subseteq W_0$ to ensure that $W = \bigcup_{i \leq |Q| \cdot r} W_i \times \{i \bmod r\}$ is closed, (2) $P_{|Q| \cdot r} = \emptyset$ showing that from every state in $W_0 \times \{0\}$ there is a path to $R \times \{0\}$ (and thus also from all states in W), and (3) the state q_{init} occurs in some slice W_i . The correctness of the algorithm follows from the characterization of the almost-sure winning set for reachability in MDPs: if some state $\langle q_{\text{init}}, t \rangle$ is limit-sure winning, then the algorithm accepts by guessing (slice by slice) the almost-sure winning set W and the paths from $W_0 \times \{0\}$ to $R \times \{0\}$ (at position 0 modulo r), and otherwise any set (and paths) correctly guessed by the algorithm would not contain q_{init} in any slice. \square

It follows from the proof of Lemma 10 that all winning modes for eventually synchronizing are independent of the numerical value of the positive transition probabilities.

Corollary 2. *Let $\mu \in \{\text{sure}, \text{almost}, \text{limit}\}$ and $T \subseteq U$ be two sets. For two distributions d, d' with $\text{Supp}(d) = \text{Supp}(d')$, we have $d \in \langle \langle 1 \rangle \rangle_\mu^{\text{event}}(\text{sum}_T, U)$ if and only if $d' \in \langle \langle 1 \rangle \rangle_\mu^{\text{event}}(\text{sum}_T, U)$.*

To establish the PSPACE-hardness for limit-sure eventually synchronizing in MDPs, we use a reduction from the universal finiteness problem for 1L-AFAs.

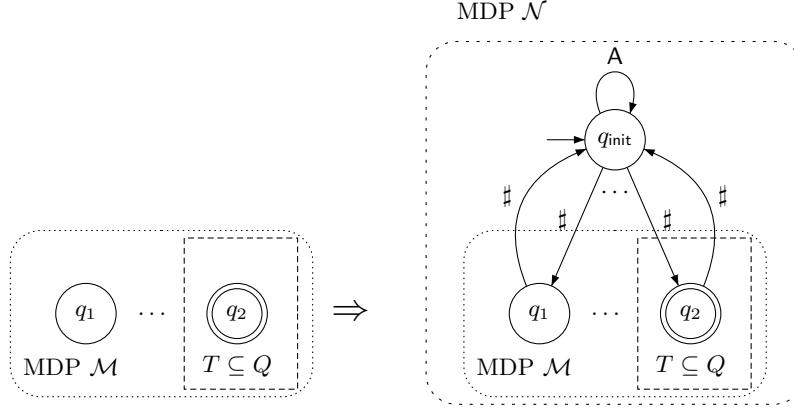


Fig. 8. Sketch of the reduction to show PSPACE-hardness of the membership problem for limit-sure eventually and almost-sure weakly synchronizing.

Lemma 12. *The membership problem for $\langle\langle 1 \rangle\rangle_{limit}^{event}(sum_T)$ is PSPACE-hard even if T is a singleton.*

Proof. We show the result by a reduction from the universal finiteness problem for one-letter alternating automata (1L-AFA), which is PSPACE-complete (by Lemma 4). It is easy to see that this problem remains PSPACE-complete even if the set T of accepting states of the 1L-AFA is a singleton, and given the tight relation between 1L-AFA and MDP (see Section 2.4), it follows from the definition of the universal finiteness problem that deciding, in an MDP \mathcal{M} , whether the sequence $Pre_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$ is PSPACE-complete.

The reduction is as follows (see also Fig. 8). Given an MDP $\mathcal{M} = \langle Q, A, \delta \rangle$ and a singleton $T \subseteq Q$, we construct an MDP $\mathcal{N} = \langle Q', A', \delta' \rangle$ with state space $Q' = Q \uplus \{q_{init}\}$ such that $Pre_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$ if and only if q_{init} is limit-sure eventually synchronizing in T . The MDP \mathcal{N} is essentially a copy of \mathcal{M} with alphabet $A \uplus \{\#\}$ and the transition function on action $\#$ is the uniform distribution on Q from q_{init} , and the Dirac distribution on q_{init} from the other states $q \in Q$. There are self-loops on q_{init} for all other actions $a \in A$. Formally, the transition function δ' is defined as follows, for all $q \in Q$:

- $\delta'(q, a) = \delta(q, a)$ for all $a \in A$ (copy of \mathcal{M}), and $\delta'(q, \#)(q_{init}) = 1$;
- $\delta'(q_{init}, a)(q_{init}) = 1$ for all $a \in A$, and $\delta'(q_{init}, \#)(q) = \frac{1}{|Q|}$.

We establish the correctness of the reduction as follows. For the first direction, assume that $Pre_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$. Then since \mathcal{N} embeds a copy of \mathcal{M} it follows that $Pre_{\mathcal{N}}^n(T) \neq \emptyset$ for all $n \geq 0$ and there exist numbers $k_0, r \leq 2^{|Q|}$ such that $Pre_{\mathcal{N}}^{k_0+r}(T) = Pre_{\mathcal{N}}^{k_0}(T) \neq \emptyset$. Using Lemma 9 with $k = k_0$ and $R = Pre_{\mathcal{N}}^{k_0}(T)$ (and $U = Z = Q'$ is the trivial support), it is sufficient to prove that $q_{init} \in \langle\langle 1 \rangle\rangle_{limit}^{event}(R)$ to get $q_{init} \in \langle\langle 1 \rangle\rangle_{limit}^{event}(T)$ (in \mathcal{N}). We show the stronger statement that q_{init} is actually almost-sure eventually synchronizing in R with the pure strategy α defined as follows, for all play prefixes ρ (let $m = |\rho| \bmod r$):

- if $Last(\rho) = q_{init}$, then $\alpha(\rho) = \#$;
- if $Last(\rho) = q \in Q$, then

- if $q \in \text{Pre}_{\mathcal{N}}^{r-m}(R)$, then $\alpha(\rho)$ plays a R -safe action at position $r - m$;
- otherwise, $\alpha(\rho) = \sharp$.

The strategy α ensures that the probability mass that is not (yet) in the sequence of predecessors $\text{Pre}_{\mathcal{N}}^n(R)$ goes to q_{init} , where by playing \sharp at least a fraction $\frac{1}{|Q|}$ of it would reach the sequence of predecessors (at a synchronized position). It follows that after $2i$ steps, the probability mass in q_{init} is $(1 - \frac{1}{|Q|})^i$ and the probability mass in the sequence of predecessors is $1 - (1 - \frac{1}{|Q|})^i$. For $i \rightarrow \infty$, the probability in the sequence of predecessors tends to 1 and since $\text{Pre}_{\mathcal{N}}^n(R) = R$ for all positions n that are a multiple of r , we get $\sup_n \mathcal{M}_n^\alpha(R) = 1$ and $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(R)$.

For the converse direction, assume that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(T)$ is limit-sure eventually synchronizing in T . By Lemma 9, either (1) q_{init} is limit-sure eventually synchronizing in $\text{Pre}_{\mathcal{N}}^n(T)$ for all $n \geq 0$, and then it follows that $\text{Pre}_{\mathcal{N}}^n(T) \neq \emptyset$ for all $n \geq 0$, or (2) q_{init} is sure eventually synchronizing in T , and then since only the action \sharp leaves the state q_{init} (and $\text{post}(q_{\text{init}}, \sharp) = Q$), the characterization of Lemma 5 shows that $Q \subseteq \text{Pre}_{\mathcal{N}}^k(T)$ for some $k \geq 0$, and since $Q \subseteq \text{Pre}_{\mathcal{N}}(Q)$ and $\text{Pre}_{\mathcal{N}}(\cdot)$ is a monotone operator, it follows that $Q \subseteq \text{Pre}_{\mathcal{N}}^n(T)$ for all $n \geq k$ and thus $\text{Pre}_{\mathcal{N}}^n(T) \neq \emptyset$ for all $n \geq 0$. We conclude the proof by noting that $\text{Pre}_{\mathcal{M}}^n(T) = \text{Pre}_{\mathcal{N}}^n(T) \cap Q$ and therefore $\text{Pre}_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$. \square

The example in the proof of Lemma 6 can be used to show that the memory needed by a family of strategies to win limit-sure eventually synchronizing objective (in target $T = \{q_2\}$) is unbounded. Observe that given $\varepsilon > 0$, the required memory to accumulate $1 - \varepsilon$ in T is finite, but the memory size increases and cannot be bounded as ε tends to 0. The following theorem summarizes the results for limit-sure eventually synchronizing.

Theorem 4. *For limit-sure eventually synchronizing (with or without exact support) in MDPs:*

1. (Complexity). *The membership problem is PSPACE-complete.*
2. (Memory). *Unbounded memory is required for both pure and randomized strategies, and pure strategies are sufficient.*

4 Weakly Synchronizing

We establish the complexity and memory requirement for weakly synchronizing objectives. We show that the membership problem is PSPACE-complete for sure and almost-sure winning, that exponential memory is necessary and sufficient for sure winning while infinite memory is necessary for almost-sure winning, and we show that limit-sure and almost-sure winning coincide. By Lemma 3, the complexity results established in this section for function sum_T hold for function max_T as well.

The weakly synchronizing objective is reminiscent of a Büchi objective in the distribution-based semantics: it requires that in the sequence of distributions of an MDP \mathcal{M} under strategy α we have $\limsup_{n \rightarrow \infty} \mathcal{M}_n^\alpha(T) = 1$ (and that $\mathcal{M}_n^\alpha(T) = 1$ for infinitely many n in the case of sure winning).

The sure winning mode can be solved by a technique similar to the search for a lasso in Büchi automata [45] (Section 4.1). We show that the almost-sure winning mode can be solved by a reduction analogous to the case of eventually synchronizing (Section 4.2). For the limit-sure winning mode, we show that it coincides with the almost-sure winning

mode. The proof of this result is technical and requires a careful characterization of the limit-sure winning mode. We present examples to provide intuitive illustration of the proof (Section 4.3).

4.1 Sure weakly synchronizing

The PSPACE upper bound of the membership problem for sure weakly synchronizing is obtained by the following characterization.

Lemma 13. *Let \mathcal{M} be an MDP and T be a target set. For all states q_{init} , we have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\text{sum}_T)$ if and only if there exists a set $S \subseteq T$ such that $q_{\text{init}} \in \text{Pre}^m(S)$ for some $m \geq 0$ and $S \subseteq \text{Pre}^n(S)$ for some $n \geq 1$.*

Proof. First, if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\text{sum}_T)$, then let α be a sure winning weakly synchronizing strategy. Then there are infinitely many positions n such that $\mathcal{M}_n^\alpha(T) = 1$, and since the state space is finite, there is a set S of states such that for infinitely many positions n we have $\text{Supp}(\mathcal{M}_n^\alpha) = S$ and $\mathcal{M}_n^\alpha(T) = 1$, and thus $S \subseteq T$. By Lemma 5, it follows that $q_{\text{init}} \in \text{Pre}^m(S)$ for some $m \geq 0$, and by considering two positions $n_1 < n_2$ where $\text{Supp}(\mathcal{M}_{n_1}^\alpha) = \text{Supp}(\mathcal{M}_{n_2}^\alpha) = S$, it follows that $S \subseteq \text{Pre}^n(S)$ for $n = n_2 - n_1 \geq 1$.

The reverse direction is straightforward by considering a strategy α that ensures $\mathcal{M}_m^\alpha(S) = 1$ for some $m \geq 0$, and then ensures that the probability mass from all states in S remains in S after every multiple of n steps where $n > 0$ is such that $S \subseteq \text{Pre}^n(S)$, showing that α is a sure winning weakly synchronizing strategy in S (and thus in T) from q_{init} , thus $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\text{sum}_T)$. \square

The PSPACE upper bound follows from the characterization in Lemma 13. A (N)PSPACE algorithm is to guess the set $S \subseteq T$, and the numbers m, n (with $m, n \leq 2^{|Q|}$ since the sequence $\text{Pre}^n(S)$ of predecessors is ultimately periodic), and check that $q_{\text{init}} \in \text{Pre}^m(S)$ and $S \subseteq \text{Pre}^n(S)$. The PSPACE lower bound follows from the PSPACE-completeness of the membership problem for sure eventually synchronizing.

Lemma 14. *The membership problem for $\langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\text{sum}_T)$ is PSPACE-hard even if T is a singleton.*

Proof. We show the result by a reduction from the membership problem for $\langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_T)$ with a singleton T , which is PSPACE-complete (Theorem 2). From an MDP $\mathcal{M} = \langle Q, A, \delta \rangle$ with initial state q_{init} and target state \hat{q} , we construct another MDP $\mathcal{N} = \langle Q', A', \delta' \rangle$ and a target state \hat{p} such that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\hat{q})$ in \mathcal{M} if and only if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\hat{p})$ in \mathcal{N} .

The MDP \mathcal{N} is a copy of \mathcal{M} with two new states \hat{p} and sink that are reachable only by a new action \sharp (see Fig. 9). Formally, $Q' = Q \cup \{\hat{p}, \text{sink}\}$ and $A' = A \cup \{\sharp\}$. The transition function δ' is defined as follows: $\delta'(q, a) = \delta(q, a)$ for all states $q \in Q$ and $a \in A$, $\delta(q, \sharp)(\text{sink}) = 1$ for all $q \in Q' \setminus \{\hat{q}\}$ and $\delta(\hat{q}, \sharp)(\hat{p}) = 1$. The state sink is absorbing and from state \hat{p} all other transitions lead to the initial state, i.e. $\delta(\text{sink}, a)(\text{sink}) = 1$ and $\delta(\hat{p}, a)(q_{\text{init}}) = 1$ for all $a \in A$.

We establish the correctness of the reduction as follows. First, if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\hat{q})$ in \mathcal{M} , then let α be a sure winning strategy in \mathcal{M} for eventually synchronizing in $\{\hat{q}\}$. A sure winning strategy in \mathcal{N} for weakly synchronizing in $\{\hat{p}\}$ is to play according to α until the whole probability mass is in \hat{q} , then play \sharp followed by some $a \in A$ to visit \hat{p}

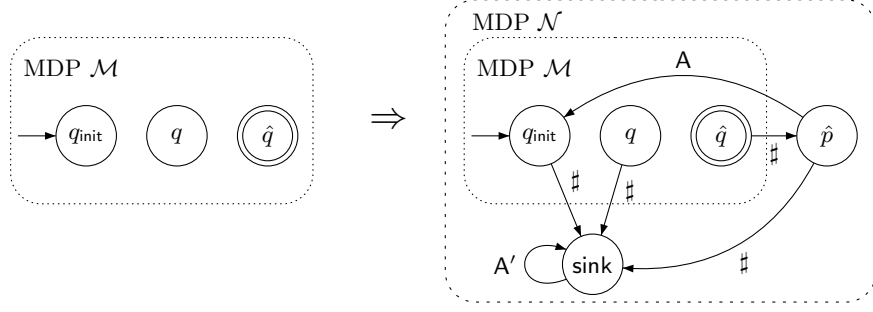


Fig. 9. The reduction sketch to show PSPACE-hardness of the membership problem for sure weakly synchronizing in MDPs.

and get back to the initial state q_{init} , and then repeat the same strategy from q_{init} . Hence $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\hat{p})$ in \mathcal{N} .

Second, if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{weakly}}(\hat{p})$ in \mathcal{N} , then consider a strategy α such that $\mathcal{N}_n^\alpha(\hat{p}) = 1$ for some $n \geq 0$. By construction of \mathcal{N} , it follows that $\mathcal{N}_{n-1}^\alpha(\hat{q}) = 1$, that is all path-outcomes of α of length $n - 1$ reach \hat{q} , and α plays \sharp in the next step. If α never plays \sharp before position $n - 1$, then α is a valid strategy in \mathcal{M} up to step $n - 1$ and it shows that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\hat{q})$ is sure winning in \mathcal{M} for eventually synchronizing in $\{\hat{q}\}$. Otherwise let m be the largest number such that there is a finite path-outcome ρ of α of length $m < n - 1$ with $\sharp \in \text{Supp}(\alpha(\rho))$. Thus between position m and $n - 1$, the strategy α does not play \sharp . Note that the action \sharp can be played by α only in the state \hat{q} , and thus $\text{Last}(\rho) = \hat{q}$. Hence two steps later, in the path-outcome ρ' of length $m + 2$ that extends ρ , we have $\text{Last}(\rho') = q_{\text{init}}$. Since the action \sharp is not played by α until position $n - 1$, after position $m + 2$ in ρ' the strategy α corresponds to a valid strategy from $\text{Last}(\rho')$ in \mathcal{M} that brings all the probability mass of $\text{Last}(\rho') = q_{\text{init}}$ to \hat{q} , witnessing that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\hat{q})$. \square

The proof of Lemma 13 suggests an exponential-memory strategy for sure weakly synchronizing that in $q \in \text{Pre}^n(S)$ plays an action a such that $\text{post}(q, a) \subseteq \text{Pre}^{n-1}(S)$, which can be realized with exponential memory since $n \leq 2^{|Q|}$. It can be shown that exponential memory is necessary in general, using an argument very similar to the proof of exponential memory lower bound for sure eventually synchronizing, and by modifying the MDPs \mathcal{M}_n presented in Section 3 (and illustrated in Fig. 5) as follows: let the transitions from state q_T go to q_{init} (instead of the absorbing state q_\perp).

Theorem 5. *For sure weakly synchronizing in MDPs:*

1. (Complexity). *The membership problem is PSPACE-complete.*
2. (Memory). *Exponential memory is necessary and sufficient for both pure and randomized strategies, and pure strategies are sufficient.*

4.2 Almost-sure weakly synchronizing

We present a characterization of almost-sure weakly synchronizing that gives a PSPACE upper bound for the membership problem. Our characterization, similar to Lemma 7, uses the limit-sure eventually synchronizing objectives with exact support introduced in

Section 3.2. We show that an MDP is almost-sure weakly synchronizing in target T if (and only if), for some set U , there is a sure eventually synchronizing strategy in target U , and from the probability distributions with support U there is a limit-sure winning strategy for eventually synchronizing in $\text{Pre}(T)$ with support in $\text{Pre}(U)$. This ensures that from the initial state we can have the whole probability mass in U , and from U have probability $1 - \varepsilon$ in $\text{Pre}(T)$ (and in T in the next step), while the whole probability mass is back in $\text{Pre}(U)$ (and in U in the next step), allowing to repeat the strategy for $\varepsilon \rightarrow 0$, thus ensuring infinitely often probability at least $1 - \varepsilon$ in T (for all $\varepsilon > 0$).

Lemma 15. *Let \mathcal{M} be an MDP and T be a target set. For all states q_{init} , we have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T)$ if and only if there exists a set U of states such that*

- $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$, and
- $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)}, \text{Pre}(U))$ where d_U is the uniform distribution over U .

Proof. First, if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T)$, then there exists a strategy α such that for all $i \geq 0$ there exists $n_i \in \mathbb{N}$ such that $\mathcal{M}_{n_i}^\alpha(T) \geq 1 - 2^{-i}$, and moreover $n_{i+1} > n_i$ for all $i \geq 0$. Let $s_i = \text{Supp}(\mathcal{M}_{n_i}^\alpha)$ be the support of $\mathcal{M}_{n_i}^\alpha$. Since the state space is finite, there is a set U that occurs infinitely often in the sequence $s_0 s_1 \dots$, thus for all $k > 0$ there exists $m_k \in \mathbb{N}$ such that $\mathcal{M}_{m_k}^\alpha(T) \geq 1 - 2^{-k}$ and $\mathcal{M}_{m_k}^\alpha(U) = 1$. It follows that α is sure eventually synchronizing in U from q_{init} , i.e. $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$. Moreover, we can assume that $m_{k+1} > m_k$ for all $k > 0$ and thus \mathcal{M} is also limit-sure eventually synchronizing in $\text{Pre}(T)$ with exact support in $\text{Pre}(U)$ from the initial distribution³ $d_1 = \mathcal{M}_{m_1}^\alpha$. Since $\text{Supp}(d_1) = U = \text{Supp}(d_U)$ and since only the support of the initial probability distributions is relevant for the limit-sure eventually synchronizing objective (Corollary 2), it follows that $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)}, \text{Pre}(U))$.

To establish the converse, note that since $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)}, \text{Pre}(U))$, it follows from Corollary 2 that from all initial distributions with support in U , for all $\varepsilon > 0$ there exists a strategy α_ε and a position n_ε such that $\mathcal{M}_{n_\varepsilon}^{\alpha_\varepsilon}(T) \geq 1 - \varepsilon$ and $\mathcal{M}_{n_\varepsilon}^{\alpha_\varepsilon}(U) = 1$. We construct an almost-sure weakly synchronizing strategy α as follows. Since $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$, play according to a sure eventually synchronizing strategy from q_{init} until all the probability mass is in U . Then for $i = 1, 2, \dots$ and $\varepsilon_i = 2^{-i}$, repeat the following procedure: given the current probability distribution, select the corresponding strategy α_{ε_i} and play according to α_{ε_i} for n_{ε_i} steps, ensuring probability mass at least $1 - 2^{-i}$ in $\text{Pre}(T)$ and support of the probability mass in $\text{Pre}(U)$. Then from states in $\text{Pre}(T)$, play an action to ensure reaching T in the next step, and from states in $\text{Pre}(U)$ ensure reaching U . Continue playing according to $\alpha_{\varepsilon_{i+1}}$ for $n_{\varepsilon_{i+1}}$ steps, etc. Since $n_{\varepsilon_i} + 1 > 0$ for all $i \geq 0$, this strategy ensures that $\limsup_{n \rightarrow \infty} \mathcal{M}_n^\alpha(T) = 1$ from q_{init} , hence $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weak}}(\text{sum}_T)$. \square

³ Note that the initial distribution $d_1 = \mathcal{M}_{m_1}^\alpha$ can be fixed before the other quantifications in the statement that we want to prove, namely: $\exists d_1 \in \mathcal{D}(U) \cdot \forall \varepsilon > 0 \cdot \exists \alpha \cdot \exists m_k : \mathcal{M}_{m_k}^\alpha(T) \geq 1 - 2^{-k}$ where we compute \mathcal{M}^α with initial distribution d_1 . This is because we fixed the strategy α in the first step of the proof, and this is why we need that q_{init} is almost-sure weakly synchronizing. Otherwise, if q_{init} is only limit-sure weakly synchronizing, we would get a possibly different initial distribution d_1 for each $\varepsilon > 0$ (induced by a possibly different strategy α for each ε). This would not always ensure that d_U is limit-sure eventually synchronizing in $\text{Pre}(T)$ with exact support in $\text{Pre}(U)$. For example if $T = \{q_1\}$ and $U = \{q_1, q_2\}$, and there are self-loops on both q_1 and q_2 (thus $\text{Pre}(T) = T$ and $\text{Pre}(U) = U$), the initial distribution defined by $d_1(q_1) = 1 - \varepsilon$ and $d_1(q_2) = \varepsilon$ has support U and ensures probability $1 - \varepsilon$ in $\text{Pre}(T)$ with support in $\text{Pre}(U)$. Thus for all $\varepsilon > 0$, we have an initial distribution that satisfies the requirements, but the distribution d_U is obviously not limit-sure eventually synchronizing in $\text{Pre}(T)$.

Since the membership problems for sure eventually synchronizing and for limit-sure eventually synchronizing with exact support are PSPACE-complete (Theorem 2 and Theorem 4), the membership problem for almost-sure weakly synchronizing is in PSPACE by guessing the set U , and checking that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_U)$, and that $d_U \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)}, \text{Pre}(U))$. We establish a matching PSPACE lower bound.

Lemma 16. *The membership problem for $\langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T)$ is PSPACE-hard even if T is a singleton.*

Proof. We use the same reduction and construction as in the PSPACE-hardness proof of Lemma 12 where from an MDP \mathcal{M} and a singleton T , we constructed N and q_{init} . Referring to that construction, we show that $\text{Pre}_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$ if and only if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(T)$.

First, if $\text{Pre}_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$, then by Lemma 15 we need to show that (i) $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_Q)$, and (ii) $d_Q \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)}, \text{Pre}(Q))$ where d_Q is the uniform distribution over Q . To show (i), we can play \sharp from q_{init} to get the probability mass synchronized in Q . To show (ii), since playing \sharp from d_Q ensures to reach q_{init} , it suffices to prove that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T, Q)$, which is done in the proof of Lemma 12.

For the converse direction, if q_{init} is almost-sure weakly synchronizing in T , then q_{init} is also limit-sure eventually synchronizing in T , and we can directly use that argument in the proof of Lemma 12 to show that $\text{Pre}_{\mathcal{M}}^n(T) \neq \emptyset$ for all $n \geq 0$.

It follows from this reduction that the membership problem for almost-sure weakly synchronization is PSPACE-hard. \square

By an argument analogous to the proof of Lemma 6, it is easy to show that in the example of Fig. 6 winning strategies require infinite memory for almost-sure weakly synchronizing.

Theorem 6. *For almost-sure weakly synchronizing in MDPs:*

1. (Complexity). *The membership problem is PSPACE-complete.*
2. (Memory). *Infinite memory is necessary in general for both pure and randomized strategies, and pure strategies are sufficient.*

4.3 Limit-sure weakly synchronizing

We show that the winning regions for almost-sure and limit-sure weakly synchronizing coincide. The result is not intuitively obvious (recall that it does not hold for eventually synchronizing) and requires a careful analysis of the structure of limit-sure winning strategies to show that they always induce the existence of an almost-sure winning strategy. The construction of an almost-sure winning strategy from a family of limit-sure winning strategies is illustrated in the following example.

Consider the MDP in Fig. 10 with initial state q_{init} and target set $T = \{q_4\}$. Note that there is a relevant strategic choice only in q_3 , and that q_{init} is limit-sure winning for eventually synchronizing in $\{q_4\}$ since we can inject a probability mass arbitrarily close to 1 in q_3 (by always playing a in q_3), and then switching to playing b in q_3 gets probability $1 - \varepsilon$ in T (for arbitrarily small ε). Moreover, the same holds from state q_4 . These two facts are sufficient to show that q_{init} is limit-sure winning for weakly synchronizing in $\{q_4\}$: given $\varepsilon > 0$, play from q_{init} a strategy to ensure probability at least $p_1 = 1 - \frac{\varepsilon}{2}$

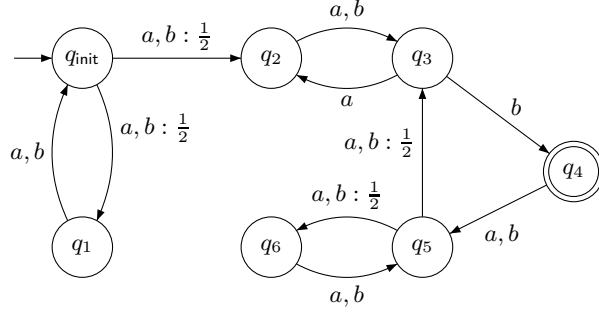


Fig. 10. An example to show $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(q_4)$ implies $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(q_4)$.

in q_4 (in finitely many steps), and then play according to a strategy that ensures from q_4 probability $p_2 = p_1 - \frac{\varepsilon}{4}$ in q_4 (in finitely many, and at least one step), and repeat this process using strategies that ensure, if the probability mass in q_4 is at least p_i , that the probability in q_4 is at least $p_{i+1} = p_i - \frac{\varepsilon}{2^{i+1}}$ (in at least one step). It follows that $p_i = 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - \dots - \frac{\varepsilon}{2^i} > 1 - \varepsilon$ for all $i \geq 1$, and thus $\limsup_{i \rightarrow \infty} p_i \geq 1 - \varepsilon$ showing that q_{init} is limit-sure weakly synchronizing in target $\{q_4\}$.

It follows from the result that we establish in this section (Theorem 7) that q_{init} is actually almost-sure weakly synchronizing in target $\{q_4\}$. To see this, consider the sequence $\text{Pre}^i(T)$ for $i \geq 0$: $\{q_4\}, \{q_3\}, \{q_2\}, \{q_3\}, \dots$ is ultimately periodic with period $r = 2$ and $R = \{q_3\} = \text{Pre}(T)$ is such that $R = \text{Pre}^2(R)$. The period corresponds to the loop q_2q_3 in the MDP. It turns out that *limit-sure* eventually synchronizing in T implies *almost-sure* eventually synchronizing in R (by the proof of Lemma 10), thus from q_{init} a *single* strategy ensures that the probability mass in R is 1, either in the limit or after finitely many steps. Note that in both cases since $R = \text{Pre}^r(R)$ this even implies almost-sure weakly synchronizing in R . The same holds from state q_4 .

Moreover, note that all distributions produced by an almost-sure weakly synchronizing strategy are themselves almost-sure weakly synchronizing. An almost-sure winning strategy for weakly synchronizing in $\{q_4\}$ consists in playing from q_{init} an *almost-sure* eventually synchronizing strategy in target $R = \{q_3\}$, and considering a decreasing sequence ε_i such that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, when the probability mass in R is at least $1 - \varepsilon_i$, inject it in $T = \{q_4\}$. Then the remaining probability mass defines a distribution (with support $\{q_1, q_2\}$ in the example) that is still almost-sure eventually synchronizing in R , as well as the states in T . Note that in the example, (almost all) the probability mass in $T = \{q_4\}$ can move to q_3 in an even number of steps, while from $\{q_1, q_2\}$ an odd number of steps is required, resulting in a *shift* of the probability mass. However, by repeating the strategy two times from q_4 (injecting large probability mass in q_3 , moving to q_4 , and injecting in q_3 again), we can make up for the shift and reach q_3 from q_4 in an even number of steps, thus in synchronization with the probability mass from $\{q_1, q_2\}$. This idea is formalized in the rest of this section, and we prove that we can always make up for the shifts, which requires a carefully analysis of the allowed amounts of shifting.

The result is easier to prove when the target T is a singleton, as in the example. For an arbitrary target set T , we need to get rid of the states in T that do not contribute a significant (i.e., bounded away from 0) probability mass in the limit, that we call the vanishing states. We show that the vanishing states can be removed from T without changing

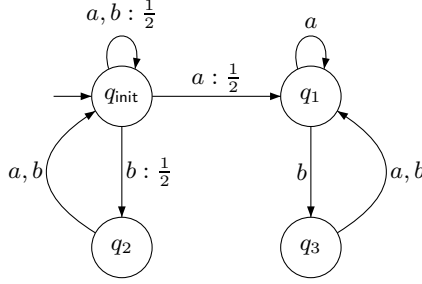


Fig. 11. The state q_2 is vanishing for target set $T = \{q_2, q_3\}$ and strategies $(\alpha_i)_{i \in \mathbb{N}}$ where α_i repeats playing i times a and then playing b , forever.

the winning region for limit-sure winning. When the target set has no vanishing state, we can construct an almost-sure winning strategy as in the case of a singleton target set.

Given an MDP \mathcal{M} with initial state $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T)$ that is limit-sure winning for the weakly synchronizing objective in target set T , let $(\alpha_i)_{i \in \mathbb{N}}$ be a family of limit-sure winning strategies such that $\limsup_{n \rightarrow \infty} \mathcal{M}_n^{\alpha_i}(T) \geq 1 - \varepsilon_i$ where $\lim_{i \rightarrow \infty} \varepsilon_i = 0$. Hence by definition of \limsup , for all $i \geq 0$ there exists a strictly increasing sequence $k_{i,0} < k_{i,1} < \dots$ of positions such that $\mathcal{M}_{k_{i,j}}^{\alpha_i}(T) \geq 1 - 2\varepsilon_i$ for all $j \geq 0$. A state $q \in T$ is *vanishing* if $\liminf_{i \rightarrow \infty} \liminf_{j \rightarrow \infty} \mathcal{M}_{k_{i,j}}^{\alpha_i}(q) = 0$ for some family of limit-sure weakly synchronizing strategies $(\alpha_i)_{i \in \mathbb{N}}$. Intuitively, the contribution of a vanishing state q to the probability in T tends to 0 and therefore \mathcal{M} is also limit-sure winning for the weakly synchronizing objective in target set $T \setminus \{q\}$.

Consider the MDP in Fig. 11 where all transitions are deterministic except from the initial state q_{init} . The state q_{init} has two successors on all actions: $\delta(q_{\text{init}}, a)(q_{\text{init}}) = \delta(q_{\text{init}}, a)(q_1) = \frac{1}{2}$ and $\delta(q_{\text{init}}, b)(q_{\text{init}}) = \delta(q_{\text{init}}, b)(q_2) = \frac{1}{2}$. Let $T = \{q_2, q_3\}$ be the target set and for all $i \in \mathbb{N}$, let α_i be the strategy that repeats forever the following template in every state: playing i times a and then playing b . The family of strategies $(\alpha_i)_{i \in \mathbb{N}}$ is a witness to show that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T)$ where the state q_2 is a vanishing state. The contribution of q_2 in accumulating the probability mass in $\{q_2, q_3\}$ tends to 0 when $i \rightarrow \infty$. As a result, $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(q_3)$ too.

Lemma 17. *If an MDP \mathcal{M} is limit-sure weakly synchronizing in target set T , then there exists a set $T' \subseteq T$ such that \mathcal{M} is limit-sure weakly synchronizing in T' without vanishing states.*

Proof. If there is no vanishing state for $(\alpha_i)_{i \in \mathbb{N}}$, then take $T' = T$ and the proof is complete. Otherwise, let $(\alpha_i)_{i \in \mathbb{N}}$ be a family of limit-sure winning strategies such that $\limsup_{n \rightarrow \infty} \mathcal{M}_n^{\alpha_i}(T) \geq 1 - \varepsilon_i$ where $\lim_{i \rightarrow \infty} \varepsilon_i = 0$ and let q be a vanishing state for $(\alpha_i)_{i \in \mathbb{N}}$. We show that $(\alpha_i)_{i \in \mathbb{N}}$ is limit-sure weakly synchronizing in $T \setminus \{q\}$. For every $i \geq 0$ let $k_{i,0} < k_{i,1} < \dots$ be a strictly increasing sequence such that (a) $\mathcal{M}_{k_{i,j}}^{\alpha_i}(T) \geq 1 - 2\varepsilon_i$ for all $i, j \geq 0$, and (b) $\liminf_{i \rightarrow \infty} \liminf_{j \rightarrow \infty} \mathcal{M}_{k_{i,j}}^{\alpha_i}(q) = 0$.

It follows from (b) that for all $\varepsilon > 0$ and all $x > 0$ there exists $i > x$ such that for all $y > 0$ there exists $j > y$ such that $\mathcal{M}_{k_{i,j}}^{\alpha_i}(q) < \varepsilon$, and thus

$$\mathcal{M}_{k_{i,j}}^{\alpha_i}(T \setminus \{q\}) \geq 1 - 2\varepsilon_i - \varepsilon$$

by (a). Since this holds for infinitely many i 's, we can choose i such that $\varepsilon_i < \varepsilon$ and we have

$$\limsup_{j \rightarrow \infty} \mathcal{M}_{k_{i,j}}^{\alpha_i}(T \setminus \{q\}) \geq 1 - 3\varepsilon$$

and thus

$$\limsup_{n \rightarrow \infty} \mathcal{M}_n^{\alpha_i}(T \setminus \{q\}) \geq 1 - 3\varepsilon$$

since the sequence $(k_{i,j})_{j \in \mathbb{N}}$ is strictly increasing. This shows that $(\alpha_i)_{i \in \mathbb{N}}$ is limit-sure weakly synchronizing in $T \setminus \{q\}$.

By repeating this argument as long as there is a vanishing state (thus at most $|T| - 1$ times), we can construct the desired set $T' \subseteq T$ without vanishing state. \square

For a limit-sure weakly synchronizing MDP in target set T (without vanishing states), we show that from a probability distribution with support T , a probability mass arbitrarily close to 1 can be injected synchronously back in T (in at least one step), that is $d_T \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$. The same holds from the initial state q_{init} of the MDP. This property is the key to construct an almost-sure weakly synchronizing strategy.

Lemma 18. *If an MDP \mathcal{M} with initial state q_{init} is limit-sure weakly synchronizing in a target set T without vanishing states, then $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$ and $d_T \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$ where d_T is the uniform distribution over T .*

Proof. Since $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T)$ and $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T) \subseteq \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T)$, we have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_T)$ and thus it suffices to prove that $d_T \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$. This is because then from q_{init} , probability arbitrarily close to 1 can be injected in $\text{Pre}(T)$ through a distribution with support in T (since by Corollary 2 only the support of the initial probability distribution is important for limit-sure eventually synchronizing).

Let $(\alpha_i)_{i \in \mathbb{N}}$ be a family of limit-sure winning strategies such that $\limsup_{n \rightarrow \infty} \mathcal{M}_n^{\alpha_i}(T) \geq 1 - \varepsilon_i$ where $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, and such that there is no vanishing state. For every $i \geq 0$ let $k_{i,0} < k_{i,1} < \dots$ be a strictly increasing sequence such that $\mathcal{M}_{k_{i,j}}^{\alpha_i}(T) \geq 1 - 2\varepsilon_i$ for all $i, j \geq 0$, and let $B = \min_{q \in T} \liminf_{i \rightarrow \infty} \liminf_{j \rightarrow \infty} \mathcal{M}_{k_{i,j}}^{\alpha_i}(q)$. Note that $B > 0$ since there is no vanishing state. It follows that there exists $x > 0$ such that for all $i > x$ there exists $y_i > 0$ such that for all $j > y_i$ and all $q \in T$ we have $\mathcal{M}_{k_{i,j}}^{\alpha_i}(q) \geq \frac{B}{2}$.

Given $\nu > 0$, let $i > x$ such that $\varepsilon_i < \frac{\nu B}{4}$, and for $j > y_i$, consider the positions $n_1 = k_{i,j}$ and $n_2 = k_{i,j+1}$. We have $n_1 < n_2$ and $\mathcal{M}_{n_1}^{\alpha_i}(T) \geq 1 - 2\varepsilon_i$ and $\mathcal{M}_{n_2}^{\alpha_i}(T) \geq 1 - 2\varepsilon_i$, and $\mathcal{M}_{n_1}^{\alpha_i}(q) \geq \frac{B}{2}$ for all $q \in T$. Consider the strategy β that plays like α_i plays from position n_1 and thus transforms the distribution $\mathcal{M}_{n_1}^{\alpha_i}$ into $\mathcal{M}_{n_2}^{\alpha_i}$. For all states $q \in T$, from the Dirac distribution on q under strategy β , the probability to reach $Q \setminus T$ in $n_2 - n_1$ steps is thus at most $\frac{\mathcal{M}_{n_2}^{\alpha_i}(Q \setminus T)}{\mathcal{M}_{n_1}^{\alpha_i}(q)} \leq \frac{2\varepsilon_i}{B/2} < \nu$.

Therefore, from an arbitrary probability distribution with support T we have $\mathcal{M}_{n_2-n_1}^{\beta}(T) > 1 - \nu$, showing that d_T is limit-sure eventually synchronizing in T and thus in $\text{Pre}(T)$ since $n_2 - n_1 > 0$ (it is easy to show that if the mass of probability in T is at least $1 - \nu$, then the mass of probability in $\text{Pre}(T)$ one step before is at least $1 - \frac{\nu}{\eta}$ where η is the smallest positive probability in \mathcal{M}). \square

To show that limit-sure and almost-sure winning coincide for weakly synchronizing objectives, from a family of limit-sure winning strategies we construct an almost-sure winning

strategy that uses the eventually synchronizing strategies of Lemma 18. The construction consists in using successively strategies that ensure probability mass $1 - \varepsilon_i$ in the target T , for a decreasing sequence $\varepsilon_i \rightarrow 0$. Such strategies exist by Lemma 18, both from the initial state and from the set T . However, the mass of probability that can be guaranteed to be synchronized in T by the successive strategies is always smaller than 1, and therefore we need to argue that the remaining mass of probability (of total size ε_i) scattered in the state space can also get synchronized in T , despite the variable shifts with the main mass of probability.

Two main key arguments are needed to establish the correctness of the construction: (1) eventually synchronizing implies that a finite number of steps is sufficient to obtain a probability mass of $1 - \varepsilon_i$ in T , and thus the construction of the strategy is well defined, and (2) by the finiteness of the period r (such that $R = \text{Pre}^r(R)$ where $R = \text{Pre}^k(T)$ for some k) we can ensure to eventually make up for the shifts, and every piece of the probability mass can eventually contribute (synchronously) to the probability accumulated in the target.

Theorem 7. $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T) = \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T)$ for all MDPs and target sets T .

Proof. Since $\langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T) \subseteq \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T)$ holds by the definition, it is sufficient to prove that $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T) \subseteq \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T)$ and by Lemma 17 it is sufficient to prove that if $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T)$ is limit-sure weakly synchronizing in T without vanishing state, then q_{init} is almost-sure weakly synchronizing in T . If T has vanishing states, then consider $T' \subseteq T$ as in Lemma 17 and it will follow that q_{init} is almost-sure weakly synchronizing in T' and thus also in T . We proceed with the proof that $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{weakly}}(\text{sum}_T)$ implies $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{weakly}}(\text{sum}_T)$.

For $i = 1, 2, \dots$ consider the sequence of predecessors $\text{Pre}^i(T)$, which is ultimately periodic: let $1 \leq k, r \leq 2^{|Q|}$ such that $\text{Pre}^k(T) = \text{Pre}^{k+r}(T)$, and let $R = \text{Pre}^k(T)$. Thus $R = \text{Pre}^{k+r}(R) = \text{Pre}^r(R)$.

Claim 1 We have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_R)$ and $d_T \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_R)$.

Proof of Claim 1 By Lemma 18, since there is no vanishing state in T we have $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$ and $d_T \in \langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$, and it follows from the characterization of Lemma 9 and Corollary 1 that:

either (1) $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$ or (2) $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_R)$, and
either (a) $d_T \in \langle\langle 1 \rangle\rangle_{\text{sure}}^{\text{event}}(\text{sum}_{\text{Pre}(T)})$ or (b) $d_T \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_R)$.

Note that (a) implies (b) (and thus (b) holds) since (a) implies $T \subseteq \text{Pre}^i(T)$ for some $i \geq 1$ (Lemma 5) and thus $T \subseteq \text{Pre}^{n \cdot i}(T)$ for all $n \geq 0$ by monotonicity of $\text{Pre}^i(\cdot)$, which entails for $n \cdot i \geq k$ that $T \subseteq \text{Pre}^m(R)$ where $m = (n \cdot i - k) \bmod r$ and thus d_T is sure (and almost-sure) winning for the eventually synchronizing objective in target R .

Note also that (1) implies (2) since by (1) we can play a sure-winning strategy from q_{init} to ensure in finitely many steps probability 1 in $\text{Pre}(T)$ and in the next step probability 1 in T , and by (b) play an almost-sure winning strategy for eventually synchronizing in R . Hence $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{event}}(\text{sum}_R)$ and thus (2b) holds, which concludes the proof of Claim 1.

We now show that there exists an almost-sure winning strategy for the weakly synchronizing objective in target T . Recall that $\text{Pre}^r(R) = R$ and thus once some probability mass p is in R , it is possible to ensure that the probability mass in R after r steps is at least p , and thus that (with period r) the probability in R does not decrease. By the result

of Lemma 10, almost-sure winning for eventually synchronizing in R implies that there exists a strategy α such that the probability in R tends to 1 at periodic positions: for some $0 \leq h < r$ the strategy α is *almost-sure eventually synchronizing in R with shift h* , that is $\forall \varepsilon > 0 \cdot \exists N \cdot \forall n \geq N : n \equiv h \pmod{r} \implies \mathcal{M}_n^\alpha(R) \geq 1 - \varepsilon$. We also say that the initial distribution $d_0 = \mathcal{M}_0^\alpha$ is almost-sure eventually synchronizing in R with shift h .

Claim 2

- (\star) If \mathcal{M}_0^α is almost-sure eventually synchronizing in R with some shift h , then \mathcal{M}_i^α is almost-sure eventually synchronizing in R with shift $h - i \pmod{r}$.
- ($\star\star$) Let t such that d_T is almost-sure eventually synchronizing in R with shift t . If a distribution is almost-sure eventually synchronizing in R with some shift h , then it is also almost-sure eventually synchronizing in R with shift $h + k + t \pmod{r}$ (where we chose k such that $R = \text{Pre}^k(T)$).

Proof of Claim 2 The result (\star) immediately follows from the definition of shift, and we prove ($\star\star$) as follows. We show that almost-sure eventually synchronizing in R with shift h implies almost-sure eventually synchronizing in R with shift $h + k + t \pmod{r}$. Intuitively, the probability mass that is in R with shift h can be injected in T in k steps, and then from T we can play an almost-sure eventually synchronizing strategy in target R with shift t , thus a total shift of $h + k + t \pmod{r}$. Precisely, an almost-sure winning strategy α is constructed as follows: given a finite prefix of play ρ , if there is no state $q \in R$ that occurs in ρ at a position $n \equiv h \pmod{r}$, then play in ρ according to the almost-sure winning strategy α_h for eventually synchronizing in R with shift h . Otherwise, if there is no $q \in T$ that occurs in ρ at a position $n \equiv h + k \pmod{r}$, then we play according to a sure winning strategy α_{sure} for eventually synchronizing in T , and otherwise we play according to an almost-sure winning strategy α_t from T for eventually synchronizing in R with shift t . To show that α is almost-sure eventually synchronizing in R with shift $h + k + t$, note that α_h ensures with probability 1 that R is reached at positions $n \equiv h \pmod{r}$, and thus T is reached at positions $h + k \pmod{r}$ by α_{sure} , and from the states in T the strategy α_t ensures with probability 1 that R is reached at positions $h + k + t \pmod{r}$. This concludes the proof of Claim 2.

Construction of an almost-sure winning strategy We construct strategies α_ε for $\varepsilon > 0$ that ensure, from a distribution that is almost-sure eventually synchronizing in R (with some shift h), that after finitely many steps, a distribution d' is reached such that $d'(T) \geq 1 - \varepsilon$ and d' is almost-sure eventually synchronizing in R (with some shift h'). Since q_{init} is almost-sure eventually synchronizing in R (with some shift h), it follows that the strategy α_{as} that plays successively the strategies (each for finitely many steps) $\alpha_{\frac{1}{2}}, \alpha_{\frac{1}{4}}, \alpha_{\frac{1}{8}}, \dots$ is almost-sure winning from q_{init} for the weakly synchronizing objective in target T .

We define the strategies α_ε as follows. Given an initial distribution that is almost-sure eventually synchronizing in R with a shift h and given $\varepsilon > 0$, let α_ε be the strategy that plays according to the almost-sure winning strategy α_h for eventually synchronizing in R with shift h for a number of steps $n \equiv h \pmod{r}$ until a distribution d is reached such that $d(R) \geq 1 - \varepsilon$, and then from d it plays according to a sure winning strategy α_{sure} for eventually synchronizing in T from the states in R (for k steps), and keeps playing according to α_h from the states in $Q \setminus R$ (for k steps). The distribution d' reached from d after k steps is such that $d'(T) \geq 1 - \varepsilon$ and we claim that it is almost-sure eventually synchronizing in R with shift t . This holds by definition from the states in $\text{Supp}(d') \cap T$,

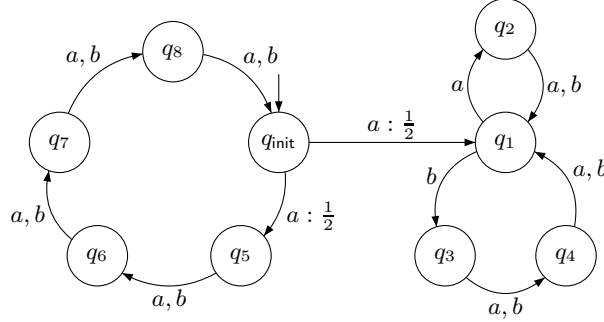


Fig. 12. An example to show $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{strong}}(\max_Q)$ reduces to synchronized reachability of a state in a simple deterministic cycle.

and by (\star) the states in $\text{Supp}(d') \setminus T$ are almost-sure eventually synchronizing in R with shift $h - (h + k) \bmod r$, and by $(\star\star)$ with shift $h - (h + k) + k + t = t$.

It follows that the strategy α_{as} is well-defined and ensures, for all $\varepsilon > 0$, that the probability mass in T is infinitely often at least $1 - \varepsilon$, thus is almost-sure weakly synchronizing in T . This concludes the proof of Theorem 7. \square

5 Strongly Synchronizing

The strongly synchronizing objective is reminiscent of a coBüchi objective in the distribution-based semantics: with function sum_T it requires that in the sequence of distributions of an MDP \mathcal{M} under strategy α we have $\liminf_{n \rightarrow \infty} \mathcal{M}_n^\alpha(T) = 1$ (and that $\mathcal{M}_n^\alpha(T) = 1$ from some point on in the case of sure winning).

We show that the membership problem for strongly synchronizing objectives can be solved in polynomial time, for all winning modes, and both with function \max_T (Section 5.1) and function sum_T (Section 5.2). We show that linear-size memory is necessary in general for \max_T , and memoryless strategies are sufficient for sum_T . It follows from our results that the limit-sure and almost-sure winning modes coincide for strongly synchronizing.

5.1 Strongly synchronizing with function \max

First, note that for strongly synchronizing the membership problem with function \max_T reduces to the membership problem with function \max_Q where Q is the entire state space, by a construction similar to the proof of Lemma 3: states in $Q \setminus T$ are duplicated, ensuring that only states in T are used to accumulate probability.

The strongly synchronizing objective with function \max_Q requires that from some point on, almost all the probability mass is at every step in a single state. Intuitively, the sequence of states that contain almost all the probability corresponds to a sequence of deterministic transitions in the MDP, and thus eventually to a cycle of deterministic transitions.

Consider the MDP in Fig. 12 with initial state q_{init} : all transitions are deterministic except from q_{init} where on both actions a and b , the successors are q_1 and q_5 with probability $\frac{1}{2}$.

The strategic choice is only relevant in q_1 where $\delta(q_1, a)(q_2) = 1$ and $\delta(q_1, b)(q_3) = 1$. We present a strategy such that the sequence of states that contain almost all the probability is the cycle $q_1 q_2 q_1 q_3 q_4 q_1$ of deterministic transitions.

The state q_{init} is almost-sure strongly synchronizing (according to function \max) with the strategy α defined as follows, for all paths ρ such that $\text{Last}(\rho) = q_1$:

- if the number of occurrences of q_1 in ρ is odd (i.e., the length of ρ is 1 modulo 5), then play action a ;
- if the number of occurrences of q_1 in ρ is even (i.e., the length of ρ is 3 modulo 5), then play action b .

The strategy α ensures the probability mass injected from q_{init} in q_1 after every other 5 steps loops in the cycle $q_1 q_2 q_1 q_3 q_4 q_1$ (with length 5). Hence the probability mass from q_{init} is always injected in q_1 synchronously (i.e., when the probability mass in the cycle is also in q_1).

It follows that after $5i$ steps, the probability mass in q_{init} is $\frac{1}{2^i}$ and the probability mass in q_1 is $1 - \frac{1}{2^i}$. Considering $i \rightarrow \infty$, we then get $\liminf_{n \rightarrow \infty} \|\mathcal{M}_n^\alpha\| = 1$ and $q_{\text{init}} \in \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{strong}}(\max)$. Note that only the states in the cycle $q_1 q_2 q_1 q_3 q_4 q_1$ (of deterministic transitions) are used to accumulate the probability mass tending to 1.

Cycles consisting of deterministic transitions are keys to decide strongly synchronizing. We define the *graph of deterministic transitions* of an MDP $\mathcal{M} = \langle Q, A, \delta \rangle$ as the directed graph $G = \langle Q, E \rangle$ where $E = \{(q_1, q_2) \mid \exists a \in A : \delta(q_1, a)(q_2) = 1\}$. For $\ell \geq 1$, a *deterministic cycle in \mathcal{M}* of length ℓ is a finite path $\hat{q}_0 \hat{q}_1 \dots \hat{q}_\ell$ in G (that is, $(\hat{q}_i, \hat{q}_{i+1}) \in E$ for all $0 \leq i < \ell$) such that $\hat{q}_0 = \hat{q}_\ell$. The cycle is *simple* if $\hat{q}_i \neq \hat{q}_j$ for all $1 \leq i < j \leq \ell$.

We show that sure (resp., almost-sure and limit-sure) strongly synchronizing is equivalent to sure (resp., almost-sure and limit-sure) reachability to a state in a *simple* deterministic cycle, with the requirement that the state can be reached in a synchronized way (i.e., by finite paths whose lengths are congruent modulo the length ℓ of the cycle).

In the MDP of Fig. 12, we can construct an almost-sure strongly synchronizing strategy β that accumulates the probability mass only in the simple cycle $q_1 q_3 q_4 q_1$. The strategy β is defined as follows, for all paths ρ such that $\text{Last}(\rho) = q_1$:

- if the length of ρ is 0 modulo 3, then play action b ;
- if the length of ρ is 1 or 2 modulo 3, then play action a .

Note that if the length of ρ is a multiple of 3 and the action b is played, then on the next visit to q_1 the length of the path is also a multiple of 3, and the action b is played again. Hence once a probability mass follows the cycle $q_1 q_3 q_4 q_1$, it will follow this cycle forever. Whenever probability mass is injected in q_1 (from q_{init}) on a path ρ of length 1 or 2 modulo 3, the action a is played to visit the other cycle $q_1 q_2 q_1$ until getting back to q_1 with a path whose length is a multiple of 3. The probability mass is then injected (synchronously) into the cycle $q_1 q_3 q_4 q_1$ where eventually the probability mass tends to 1, thus the strategy β is almost-sure strongly synchronizing and it ensures with probability 1 that q_1 is reached with by paths whose length is a multiple of 3.

We show in Lemma 19 that simple deterministic cycles are always sufficient for strongly synchronizing in MDPs, and that strongly synchronizing reduces to a synchronized reachability problem of reaching a state q_1 of a simple deterministic cycle by paths of length that is a multiple of the length ℓ of the cycle. To check synchronized reachability, we keep track of a modulo- ℓ counter along the path. Define the MDP $\mathcal{M} \times [\ell] = \langle Q', A, \delta' \rangle$ where $Q' = Q \times \{0, 1, \dots, \ell - 1\}$ and $\delta'(\langle q, i \rangle, a)(\langle q', i - 1 \rangle) = \delta(q, a)(q')$ (where $i - 1$ is $\ell - 1$ for

$i = 0$) for all states $q, q' \in Q$, actions $a \in A$, and $0 \leq i \leq \ell - 1$. Note that given a finite path $\rho = q_0 a_0 q_1 a_1 \dots q_n$ in \mathcal{M} , there is a corresponding path $\rho' = \langle q_0, k_0 \rangle a_0 \langle q_1, k_1 \rangle a_1 \dots \langle q_n, k_n \rangle$ in $\mathcal{M} \times [\ell]$ where $k_i = -i \bmod \ell$. Since the sequence $k_0 k_1 \dots$ is uniquely defined, there is a clear bijection between the paths in \mathcal{M} (starting from q_0) and the paths in $\mathcal{M} \times [\ell]$ (starting from $\langle q_0, 0 \rangle$) that we often omit to apply and mention in the sequel.

Lemma 19. *Let η be the smallest positive probability in the transitions of \mathcal{M} , and let $\frac{1}{1+\eta} < p \leq 1$. There exists a strategy α such that $\liminf_{n \rightarrow \infty} \|\mathcal{M}_n^\alpha\| \geq p$ from an initial state q_{init} if and only if there exist a simple deterministic cycle $\hat{q}_0 \hat{q}_1 \dots \hat{q}_\ell$ in \mathcal{M} and a strategy β in $\mathcal{M} \times [\ell]$ such that $\Pr^\beta(\diamond\{\langle \hat{q}_0, 0 \rangle\}) \geq p$ from $\langle q_{\text{init}}, 0 \rangle$.*

Proof. For the first direction of the lemma, assume that there exists a strategy α such that $\liminf_{n \rightarrow \infty} \|\mathcal{M}_n^\alpha\| \geq p$ from q_{init} . Thus for all $\varepsilon > 0$ (in particular, we consider $\varepsilon < p - \frac{1}{1+\eta}$), there exists $k \in \mathbb{N}$ such that for all $n \geq k$ we have $\|\mathcal{M}_n^\alpha\| \geq p - \varepsilon$, and let \hat{p}_n be a state such that $\mathcal{M}_n^\alpha(\hat{p}_n) \geq p - \varepsilon$. We claim that for all $n \geq k$, there exists an action $a \in A$ such that $\text{post}(\hat{p}_n, a) = \{\hat{p}_{n+1}\}$ i.e., there is a deterministic transition from \hat{p}_n to \hat{p}_{n+1} . Assume towards contradiction that for some $n \geq k$, for all $a \in A$ there exists $q_a \neq \hat{p}_{n+1}$ such that $q_a \in \text{post}(\hat{p}_n, a)$. Then no matter the actions played by α at step n , we have $\mathcal{M}_{n+1}^\alpha(\{q_a \mid a \in A\}) \geq \mathcal{M}_n^\alpha(\hat{p}_n) \cdot \eta \geq (p - \varepsilon) \cdot \eta$, and since $\hat{p}_{n+1} \neq q_a$ for all $a \in A$, it follows that

$$\mathcal{M}_{n+1}^\alpha(\hat{p}_{n+1}) \leq 1 - \mathcal{M}_{n+1}^\alpha(\{q_a \mid a \in A\}) \leq 1 - (p - \varepsilon) \cdot \eta \leq 1 - \frac{\eta}{1+\eta} < p - \varepsilon,$$

in contradiction with the fact that \hat{p}_{n+1} is a state such that $\mathcal{M}_{n+1}^\alpha(\hat{p}_{n+1}) \geq p - \varepsilon$. This concludes the argument showing that for all $n \geq k$, there exists an action $a \in A$ such that $\text{post}(\hat{p}_n, a) = \{\hat{p}_{n+1}\}$.

Now in the sequence $\hat{p}_k \hat{p}_{k+1} \dots$, we can extract a simple (and deterministic) cycle $\mathcal{C} = \hat{p}_i \hat{p}_{i+1} \dots \hat{p}_{i+\ell}$ since the state space is finite. Let $\hat{q}_0 = \hat{p}_{i+j}$ where $j \leq \ell$ is such that $i+j \bmod \ell = 0$. Then \hat{q}_0 is on a simple deterministic cycle, and is reachable after a multiple of ℓ steps with probability at least $p - \varepsilon$ by a strategy β in $\mathcal{M} \times [\ell]$ that copies the strategy α . Hence we have $\Pr^\beta(\diamond\{\langle \hat{q}_0, 0 \rangle\}) \geq p - \varepsilon$ from $\langle q_{\text{init}}, 0 \rangle$. Since for every $\varepsilon > 0$, we can find such a cycle and state \hat{q}_0 , and since the state space is finite (as well as the number of simple cycles), it follows that there is a cycle \mathcal{C} and state \hat{q}_0 in \mathcal{C} such that for all $\varepsilon > 0$ we have $\Pr^\beta(\diamond\{\langle \hat{q}_0, 0 \rangle\}) \geq p - \varepsilon$, and thus $\Pr^\beta(\diamond\{\langle \hat{q}_0, 0 \rangle\}) \geq p$.

For the second direction of the lemma, assume that there exist a simple deterministic cycle $\hat{q}_0 \hat{q}_1 \dots \hat{q}_\ell$ and a strategy β in $\mathcal{M} \times [\ell]$ that ensures the target set $\{\langle \hat{q}_0, 0 \rangle\}$ is reached with probability at least p from $\langle q_{\text{init}}, 0 \rangle$. Since randomization is not necessary for reachability objectives in MDPs, we can assume that β is a pure strategy. We show that there exists a strategy α such that $\liminf_{n \rightarrow \infty} \|\mathcal{M}_n^\alpha\| \geq p$ from q_{init} . From β , we construct a pure strategy α in \mathcal{M} . Given $\rho = q_0 a_0 q_1 a_1 \dots q_n$, we define $\alpha(\rho)$ as follows: if $q_n = \hat{q}_{n \bmod \ell}$, then there exists an action a such that $\text{post}(q_n, a) = \{\hat{q}_{n+1 \bmod \ell}\}$ and we define $\alpha(\rho) = a$, otherwise let $\alpha(\rho) = \beta(\rho)$. Thus α mimics β until a state \hat{q}_k of the cycle is reached at step n such that $k = n \bmod \ell$, and then α switches to always playing actions that keeps \mathcal{M} in the simple deterministic cycle $\hat{q}_0 \hat{q}_1 \dots \hat{q}_\ell$.

We claim that given $\varepsilon > 0$ there exists k such that for all $n \geq k$, we have $\|\mathcal{M}_n^\alpha\| \geq p - \varepsilon$, which entails that $\liminf_{n \rightarrow \infty} \|\mathcal{M}_n^\alpha\| \geq p$ from q_{init} and concludes the proof. To show the claim, since $\Pr^\beta(\diamond\{\langle \hat{q}_0, 0 \rangle\}) \geq p$, consider k such that $\Pr^\beta(\diamond^{\leq k}\{\langle \hat{q}_0, 0 \rangle\}) \geq p - \varepsilon$,

and for $i = 1, 2, \dots, \ell$, let $R_i = \{\langle \hat{q}_i, \ell - i \rangle\}$. Note that $R_\ell = \{\langle \hat{q}_0, 0 \rangle\}$. Then trivially $\Pr^\beta(\diamond^{\leq k} \bigcup_{i=1}^\ell R_i) \geq p - \varepsilon$ and since α agrees with β on all finite paths that do not (yet)

visit $\bigcup_{i=1}^{\ell} R_i$, given a path ρ that visits $\bigcup_{i=1}^{\ell} R_i$ (for the first time), only actions that keep \mathcal{M} in the simple cycle $\hat{q}_0\hat{q}_1\ldots\hat{q}_\ell$ are played by α and thus all continuations of ρ in the outcome of α will visit \hat{q}_0 after a multiple of ℓ steps, say $j \cdot \ell$ steps (in total). Since next, α will always play actions that keeps \mathcal{M} looping through the cycle $\hat{q}_0\hat{q}_1\ldots\hat{q}_\ell$, we have $\mathcal{M}_{j \cdot \ell + i}^\alpha(\hat{q}_i) \geq p - \varepsilon$ for all $0 \leq i < \ell$, and thus $\|\mathcal{M}_n^\alpha\| \geq p - \varepsilon$ for all $n \geq j \cdot \ell$. \square

It follows directly from Lemma 19 with $p = 1$ that almost-sure strongly synchronizing is equivalent to almost-sure reachability to a deterministic cycle in $\mathcal{M} \times [\ell]$. The same equivalence holds for the sure and limit-sure winning modes.

Lemma 20. *A state q_{init} is sure (resp., almost-sure or limit-sure) winning for the strongly synchronizing objective (according to \max_Q) in \mathcal{M} if and only if there exists a simple deterministic cycle $\hat{q}_0\hat{q}_1\ldots\hat{q}_\ell$ such that $\langle q_{\text{init}}, 0 \rangle$ is sure (resp., almost-sure or limit-sure) winning for the reachability objective $\diamond\{\langle \hat{q}_0, 0 \rangle\}$ in $\mathcal{M} \times [\ell]$.*

Proof. We consider the three winning modes:

(1) sure winning mode. The proof is similar to the proof of Lemma 19. For the first direction, given a strategy α and k such that for all $n \geq k$ we have $\|\mathcal{M}_n^\alpha\| = 1$ from the initial state q_{init} , we can construct a sequence $\hat{p}_k\hat{p}_{k+1}\ldots$ of states where there is deterministic transition from \hat{p}_n to \hat{p}_{n+1} for all $n \geq k$ (let \hat{p}_n be the state such that $\mathcal{M}_n^\alpha(\hat{p}_n) = 1$). This sequence contains a simple deterministic cycle and a state \hat{q}_0 in this cycle occurs in the sequence at a position $\hat{p}_{j \cdot \ell}$ that is a multiple of the length ℓ of the cycle. Hence the strategy α played in $\mathcal{M} \times [\ell]$ ensures to reach $\langle \hat{q}_0, 0 \rangle$ surely from $\langle q_{\text{init}}, 0 \rangle$.

For the second direction, if a strategy β ensures to reach a state $\langle \hat{q}_0, 0 \rangle$ in $\mathcal{M} \times [\ell]$ where \hat{q}_0 belongs to a simple deterministic cycle of length ℓ , then a strategy α that mimics β until $\langle \hat{q}_0, 0 \rangle$ is reached, and then switches to playing actions to follow the simple cycle, ensures sure strongly synchronizing with function \max_Q in \mathcal{M} .

(2) almost-sure winning mode. This case follows from Lemma 19 with $p = 1$.

(3) limit-sure winning mode: For the first direction, if q_{init} is limit-sure winning for the strongly synchronizing objective, then for all $\varepsilon > 0$, there exists a strategy α such that $\liminf_{n \rightarrow \infty} \|\mathcal{M}_n^{\alpha_i}\| \geq 1 - \varepsilon$. By Lemma 19, for a decreasing sequence $\varepsilon_i \rightarrow 0$ such that $\varepsilon_i < 1 - \frac{1}{1+\eta}$ there exist a simple deterministic cycle \mathcal{C}_i of length ℓ_i , a state \hat{q}_0^i in \mathcal{C}_i , and a strategy β_i in $\mathcal{M} \times [\ell_i]$ such that $\Pr^{\beta_i}(\diamond\{\langle \hat{q}_0^i, 0 \rangle\}) \geq 1 - \varepsilon_i$ from $\langle q_{\text{init}}, 0 \rangle$. Since there is a finite number of simple deterministic cycles in \mathcal{M} , some simple cycle $\mathcal{C} = \hat{q}_0\hat{q}_1\ldots\hat{q}_\ell$ and state \hat{q}_0 occurs infinitely often in the sequence of $(\mathcal{C}_i, \hat{q}_0^i)$, and thus $\langle q_{\text{init}}, 0 \rangle$ is limit-sure winning for the reachability objective $\diamond\{\langle \hat{q}_0, 0 \rangle\}$ in $\mathcal{M} \times [\ell]$.

For the second direction, since limit-sure winning implies almost-sure winning for reachability objectives in MDPs, it follows from case **(2)** that q_{init} is almost-sure (and thus also limit-sure) winning for the strongly synchronizing objective in \mathcal{M} . \square

Since the winning regions of almost-sure and limit-sure winning coincide for reachability objectives in MDPs [18], the next corollary follows from Lemma 20.

Corollary 3. $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{strongly}}(\max_T) = \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{strongly}}(\max_T)$ for all target sets T .

If there exists a cycle \mathcal{C} satisfying the condition in Lemma 20, then all cycles reachable from \mathcal{C} in the graph G of deterministic transitions also satisfies the condition. Hence it is sufficient to check the condition for an arbitrary simple cycle in each strongly connected component (SCC) of G . As shown in the next theorem, it follows that strongly synchronizing can be decided in polynomial time and the length of the cycle gives a linear bound on the memory needed to win.

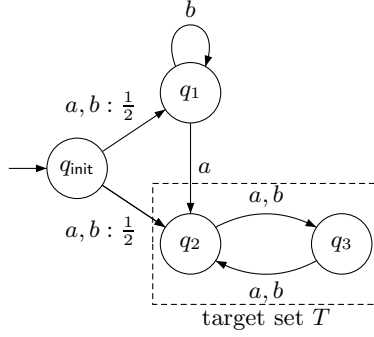


Fig. 13. An MDP where all strategies to win sure strongly synchronizing with function $\max_{\{q_2, q_3\}}$ require memory.

Theorem 8. *For the three winning modes of strongly synchronizing according to \max_T :*

1. (Complexity). *The membership problem is PTIME-complete.*
2. (Memory). *Linear memory is necessary and sufficient for both pure and randomized strategies, and pure strategies are sufficient.*

Proof. First, we prove the PTIME upper bound. Given an MDP $\mathcal{M} = \langle Q, A, \delta \rangle$ and a state q_{init} , we say that a simple deterministic cycle $\mathcal{C} = \hat{q}_0 \hat{q}_1 \dots \hat{q}_\ell$ is sure (resp., almost-sure, and limit-sure) winning from q_{init} if $\langle q_{\text{init}}, 0 \rangle$ is sure (resp., almost-sure, and limit-sure) winning for the reachability objective $\Diamond \{ \langle \hat{q}_0, 0 \rangle \}$ in $\mathcal{M} \times [\ell]$.

We claim that if \mathcal{C} is sure (resp., almost-sure, and limit-sure) winning from q_{init} , then so are all simple cycles \mathcal{C}' reachable from \mathcal{C} in the graph of deterministic transitions induced by \mathcal{M} . Given a strategy to reach a state \hat{q}_0 of \mathcal{C} surely (resp., with probability p), we can use the path of deterministic transitions from \mathcal{C} to \mathcal{C}' to obtain a strategy to reach a state \hat{q}'_0 of \mathcal{C}' surely (resp., with probability p): since \hat{q}_0 is reached after a multiple of ℓ steps ($\{ \langle \hat{q}_0, 0 \rangle \}$ is reached in $\mathcal{M} \times [\ell]$), we can let the probability mass loop through the cycle \mathcal{C} , and transfer it to \mathcal{C}' after a number of steps that is also a multiple of ℓ' , and then let it loop in \mathcal{C}' , ensuring that $\langle \hat{q}'_0, 0 \rangle$ is reached surely (resp., with probability p) in $\mathcal{M} \times [\ell']$. This establishes the claim for the three winning modes.

Using this claim and Lemma 20, it suffices to decide sure (resp., almost-sure, and limit-sure) winning for one simple cycle in each bottom SCC (reachable from q_{init}) of the graph of deterministic transitions. Since SCC decomposition for graphs, as well as sure, almost-sure, and limit-sure reachability for MDPs can be computed in polynomial time, and the number of bottom SCCs is at most the size $|Q|$ of the graph, the PTIME upper bound for the membership problem follows.

For PTIME-hardness, the proof is by a reduction from the monotone Boolean circuit value problem, which is PTIME-complete [28]. This problem is to compute the output value of a given Boolean circuit consisting of AND-gates, OR-gates, and fixed Boolean input values. From a circuit, we construct an MDP \mathcal{M} with actions L and R , where the states correspond to the gates and input values of the circuit, and with three new absorbing states q_1 , q_2 , and sync . The successors of an AND-gate $n_1 \wedge n_2$ are n_1 and n_2 with probability $\frac{1}{2}$ on all actions, the successors of an OR-gate $n_1 \vee n_2$ are n_1 on action L , and n_2 on action R . On all actions, a node defining input value 1 has unique successor sync ,

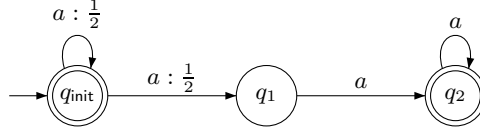


Fig. 14. An MDP such that q_{init} is sure-winning for coBüchi objective in $T = \{q_{\text{init}}, q_2\}$ but not for strongly synchronizing according to sum_T .

and a node defining input value 0 has successors q_1 and q_2 with probability $\frac{1}{2}$. Let q_{init} be the state corresponding to the output node. Then \mathcal{M} is sure (resp., almost-sure, limit-sure) strongly synchronizing (in sync) from q_{init} if and only if the value of the circuit is 1, which establishes PTIME-hardness of strongly synchronizing in the three winning modes.

Finally, the result on memory requirement is established as follows. Since memoryless strategies are sufficient for reachability objectives in MDPs, it follows from the proof of Lemma 19 and Lemma 20 that the (memoryless) winning strategies in $\mathcal{M} \times [\ell]$ can be transferred to winning strategies with memory $\{0, 1, \dots, \ell - 1\}$ in \mathcal{M} . Since $\ell \leq |Q|$, linear-size memory is sufficient to win strongly synchronizing objectives. We present a family of MDPs \mathcal{M}_n ($n \in \mathbb{N}$) that are sure winning for strongly synchronizing (according to max_Q), and where the sure winning strategies require linear memory. The MDP \mathcal{M}_2 is shown in Fig. 13, and the MDP \mathcal{M}_n is obtained from \mathcal{M}_2 by replacing the cycle q_2q_3 of deterministic transitions by a simple cycle of length n . Note that only in q_1 there is a relevant strategic choice. Since both q_1 and q_2 contain probability mass after one step, we need to wait in q_1 (by playing b) until the probability mass in q_2 comes back to q_2 through the cycle. It is easy to show that to ensure strongly synchronizing, we need to play $n - 1$ times b in q_1 before playing a , and this requires linear memory. \square

5.2 Strongly synchronizing with function sum

The strongly synchronizing objective with function sum_T requires that eventually all the probability mass remains in T . We show that this is equivalent to a traditional reachability objective with target defined by the set S of sure winning initial distributions for the safety objective $\square T$.

It follows that almost-sure (and limit-sure) winning for strongly synchronizing is equivalent to almost-sure (or equivalently limit-sure) winning for the coBüchi objective $\diamond \square T = \{q_0 a_0 q_1 \dots \in \text{Path}(\mathcal{M}) \mid \exists j \cdot \forall i > j : q_i \in T\}$ in the state-based semantics. However, sure strongly synchronizing is not equivalent to sure winning for the coBüchi objective: the MDP in Fig. 14 is sure winning for the coBüchi objective $\diamond \square \{q_{\text{init}}, q_2\}$ from q_{init} , but not sure winning for the reachability objective $\diamond S$ where $S = \{q_2\}$ is the winning region for the safety objective $\square \{q_{\text{init}}, q_2\}$ (and thus not sure strongly synchronizing). Note that this MDP is almost-sure strongly synchronizing in target $T = \{q_{\text{init}}, q_2\}$ from q_{init} , and almost-sure winning for the coBüchi objective $\diamond \square T$, as well as almost-sure winning for the reachability objective $\diamond S$.

Lemma 21. *Given a target set T , an MDP \mathcal{M} is sure (resp., almost-sure or limit-sure) winning for the strongly synchronizing objective according to sum_T if and only if \mathcal{M} is sure (resp., almost-sure or limit-sure) winning for the reachability objective $\diamond S$ where S is the sure winning region for the safety objective $\square T$.*

Proof. First, assume that a state q_{init} of \mathcal{M} is sure (resp., almost-sure or limit-sure) winning for the strongly synchronizing objective according to sum_T , and show that q_{init} is sure (resp., almost-sure or limit-sure) winning for the reachability objective $\Diamond S$.

(i) *Limit-sure winning.* For all $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{|Q|} \cdot \eta^{|Q|}$ where η is the smallest positive probability in the transitions of \mathcal{M} . By the assumption, from q_{init} there exists a strategy α and $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\mathcal{M}_n^\alpha(T) \geq 1 - \varepsilon'$. We claim that at step N , all non-safe states have probability at most $\frac{\varepsilon}{|Q|}$, that is $\mathcal{M}_N^\alpha(q) \leq \frac{\varepsilon}{|Q|}$ for all $q \in Q \setminus S$. Towards contradiction, assume that $\mathcal{M}_N^\alpha(q) > \frac{\varepsilon}{|Q|}$ for some non-safe state $q \in Q \setminus S$. Since $q \notin S$ is not safe, there is a path of length $\ell \leq |Q|$ from q to a state in $Q \setminus T$, thus with probability at least $\eta^{|Q|}$. It follows that after $N + \ell$ steps we have $\mathcal{M}_{N+\ell}^\alpha(Q \setminus T) > \frac{\varepsilon}{|Q|} \cdot \eta^{|Q|} = \varepsilon'$, in contradiction with the fact $\mathcal{M}_n^\alpha(T) \geq 1 - \varepsilon'$ for all $n \geq N$. Now, since all non-safe states have probability at most $\frac{\varepsilon}{|Q|}$ at step N , it follows that $\mathcal{M}_N^\alpha(Q \setminus S) \leq \frac{\varepsilon}{|Q|} \cdot |Q| = \varepsilon$ and thus $\Pr^\alpha(\Diamond S) \geq 1 - \varepsilon$. Therefore \mathcal{M} is limit-sure winning for the reachability objective $\Diamond S$ from q_{init} .

(ii) *Almost-sure winning.* Since almost-sure strongly synchronizing implies limit-sure strongly synchronizing, it follows from (i) that \mathcal{M} is limit-sure (and thus also almost-sure) winning for the reachability objective $\Diamond S$, as limit-sure and almost-sure reachability coincide for MDPs [18].

(iii) *Sure winning.* From q_{init} there exists a strategy α and $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\mathcal{M}_n^\alpha(T) = 1$. Hence α is sure winning for the reachability objective $\Diamond \text{Supp}(\mathcal{M}_N^\alpha)$, and from all states in $\text{Supp}(\mathcal{M}_N^\alpha)$ the strategy α ensures that only states in T are visited. It follows that $\text{Supp}(\mathcal{M}_N^\alpha) \subseteq S$ is sure winning for the safety objective $\Box T$, and thus α is sure winning for the reachability objective $\Diamond S$ from q_{init} .

For the converse direction of the lemma, assume that a state q_{init} is sure (resp., almost-sure or limit-sure) winning for the reachability objective $\Diamond S$. We construct a winning strategy for strongly synchronizing in T as follows: play according to a sure (resp., almost-sure or limit-sure) winning strategy for the reachability objective $\Diamond S$, and whenever a state of S is reached, then switch to a winning strategy for the safety objective $\Box T$. The constructed strategy is sure (resp., almost-sure or limit-sure) winning for strongly synchronizing according to sum_T because for sure winning, after finitely many steps all paths from q_{init} end up in $S \subseteq T$ and stay in S forever, and for almost-sure (or equivalently limit-sure) winning, for all $\varepsilon > 0$, after sufficiently many steps, the set S is reached with probability at least $1 - \varepsilon$, showing that the outcome is strongly $(1 - \varepsilon)$ -synchronizing in $S \subseteq T$, thus the strategy is almost-sure (and also limit-sure) strongly synchronizing. \square

Corollary 4. $\langle\langle 1 \rangle\rangle_{\text{limit}}^{\text{strongly}}(\text{sum}_T) = \langle\langle 1 \rangle\rangle_{\text{almost}}^{\text{strongly}}(\text{sum}_T)$ for all target sets T .

The following result follows from Lemma 21 and the fact that the winning region for sure safety, sure reachability, and almost-sure reachability can be computed in polynomial time for MDPs [18]. Moreover, memoryless strategies are sufficient for these objectives.

Theorem 9. For the three winning modes of strongly synchronizing according to sum_T in MDPs:

1. (Complexity). The membership problem is PTIME-complete.
2. (Memory). Pure memoryless strategies are sufficient.

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